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FORMULATIONS OF THE EQUATIONS OF MOTION OF AN ELASTIC AIRCRAFT FOR STABILITY AND CONTROL AND FLIGHT CONTROL APPLICATIONS

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Abstract

This Technical Memorandum presents a derivation of the equations of motion for large and small disturbance perturbations from a reference state of motion. The small perturbation equations of motion are then generalized to include the effects of atmospheric turbulence and gusts on the controls-free elastic aircraft. The resulting equations are termed the EXACT formulation. These EXACT equations may be used to evaluate the stability and performance of integrated flight control systems of the Control Configured Vehicle (CCV) type. The equations are unique in that they describe a highly damped system using motion coordinates referenced to a body fixed, "mean", non-inertial axis.

The EXACT formulation of the equations of motion are difficult to solve numerically. This difficulty can be overcome in some flight control analyses by using a simplified formulation such as:

- QUASI STATIC
- MODAL SUBSTITUTION
- RESIDUAL STIFFNESS
- RESIDUAL FLEXIBILITY
- MODAL TRUNCATION

The range of the applicability of each of the specialized formulations is limited by the assumptions required to reduce the EXACT formulation to that specialized formulation. A discussion of the assumptions is presented to guide the application of each formulation to military and commercial aircraft.

It is concluded that the passive acceptance of the MODAL TRUNCATION and QUASI STATIC formulations by flight control analysts should be justified numerically at the critical flight control design points. In addition, it is concluded that the full implementation of the integrated flight control benefits, promised by the CCV concept, has a current limitation due to the possible inappropriateness of the "invacuum normal modes" as elastic coordinates in a highly damped system. This limitation may be removed by developing a computational method for the accurate solution of the precise EXACT formulation.

Foreword

This Technical Memorandum was prepared under work unit 82191304. The work unit, entitled, "Analysis Methods for Control Configured Military Vehicles", is divided into five parts:

1. A literature search.

2. A formulation of the equations of motion for the controls-free aircraft.

3. A simplification of the equations in item (2) to permit a more rapid, but less accurate solution for flight control application.

4. Solutions of the equations in items (2) and (3) using USAF aircraft as test cases to develop criteria for the selection of appropriate formulation of the equations of motion.

5. A final summary report.

The work reported in this TM summarizes the results for items (1), (2), and (3) above.

The responsibility for the accuracy and conclusions presented in this Technical Memorandum rests with the organization that prepared it. This Technical Memorandum has been reviewed and is approved.

C. B. Westbrook

C. B. WESTBROOK, CHIEF
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LIST OF MATRIX SYMBOLS

Matrix Symbol	Units	Definition
A_1	lbs/v_p	Aerodynamic forces on each mass due to rigid body velocity, v_p .
A_2	lbs/$\dot{v}_p$	Aerodynamic forces on each mass due to rigid body acceleration, $\dot{v}_p$.
A_3	lbs/d_p	Aerodynamic forces on each mass due to elastic deformation position.
A_4	lbs/$\dot{d}_p$	Aerodynamic forces on each mass due to elastic deformation rate.
A_5	lbs/$\ddot{d}_p$	Aerodynamic forces on each mass due to elastic deformation acceleration.
A_{6i}	lbs/δ_i	Aerodynamic forces on each mass due to position of ith control surface, δ_i .
A_{7i}	lbs/$\dot{\delta}_i$	Aerodynamic forces on each mass due to velocity of ith control surface, $\dot{\delta}_i$.
A_{8i}	lbs/$\ddot{\delta}_i$	Aerodynamic forces on each mass due to acceleration of ith control surface, $\ddot{\delta}_i$.
C	d_p/lb	Flexibility matrix of structure cantilevered at the c.g. of the aircraft.
$\bar{C}$	d_p/lb	Flexibility matrix of the structure for a "free-free" aircraft.
d_p	ft	Displacement of mass relative to its initial position as measured relative to the body-fixed mean axis.
D	lbs/$\dot{d}_p$	Structural damping.
f	lbs	Aerodynamic forces on each mass due to rigid body and elastic motion.
f_c	lbs	Aerodynamic forces on each mass due to the controls $\delta_i(t)$.

Matrix Symbol	Units	Definition
f_g	lbs	Aerodynamic forces on each mass due to gusts.
f_t	lbs	Aerodynamic forces on each mass due to turbulence.
I_n	slugs/ft^4	Inertia of reference shape of the aircraft with respect to mean axis.
K	lbs/δ_p	Stiffness matrix of aircraft represented as "free-free" structure.
k	lbs/u	Generalized stiffness matrix.
k_1	lbs/u_1	Generalized stiffness matrix associated with retained modes r in number.
k_2	lbs/u_2	Generalized stiffness matrix associated with deleted modes.
$\underline{M}$	$\sim$	Combined rigid body inertia, $\underline{M} = \begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix}$
M	slugs	Total mass of aircraft.
M_1	ft/sec	Cross product terms in rigid body equation due to initial conditions.
M_{12}^1	ft/sec	Upper right partition of M_1 .
M_2	ft/sec^2	Gravity terms in force equations due to initial conditions.
M_{12}^2	ft/sec^2	Upper right partition of M_2 .
m_i	slugs	Mass distribution.
m	slugs	Generalized mass.
m_1	slugs	Generalized mass associated with retained modes r in number.
m_2	slugs	Generalized mass associated with deleted modes.
ρ_{ji}	ft	Equal to δ_p .

Matrix Symbol	Unit	Definition
r_{op}	none	Body axis orientation.
R	none	Residual flexibility static aeroelastic correction factor.
u	ft	Generalized coordinate defined as in-vacuum normalized mode coordinate.
u_1	ft	Generalized coordinate defined as in-vacuum normalized mode coordinate for r retained modes.
u_2	ft	Generalized coordinate defined as in-vacuum normalized mode for deleted modes.
v	ft/sec	Body axis translation rate.
v_p	$\sim$	Combination of $\dot{r}_{op}$ and v, $v_p^T = \lfloor v^T \dot{r}_{op}^T \rfloor$
$\bar{\Phi}$	$\sim$	Rigid body mode shape, $\bar{\Phi}^T = \bar{\Phi}_v^T \bar{\Phi}_r^T$
$\bar{\Phi}_v$	none	Rigid body translation mode shape.
$\bar{\Phi}_r$	ft	Rigid body rotation mode shape.
ϕ	none	Mode shapes of invacuum normalized modes, $\phi = \phi_1 \phi_2$.
ϕ_1	none	Mode shapes of retained invacuum normalized modes r in number.
ϕ_2	none	Mode shapes of deleted invacuum normalized modes.
δ_i	degrees	Motion of ith control surface.

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1.0 INTRODUCTION

Recently the flight control staffs of many aerospace research and development organizations have proposed the application of modern flight control concepts early in the design cycle of elastic aircraft. The XB-70[1] has a gust load alleviation system. The SST[2] had a "hardened" stability augmentation system to compensate the longitudinal static stability at an unstable flight condition. The B-1[3] is being designed with a "ride quality" augmentation system to ensure that the pilot can effectively monitor and control the B-1 when flying through atmospheric turbulence. Looking to the future, the USAF has proposed[4] the Control Configured Vehicle (CCV) concept to achieve full benefit of integrated flight control system technology. This concept requires, among others, that (1) the reliability of a control system be improved to a level equal to or higher than the reliability of the primary structure and (2) the usually adverse effects of the non-linearities in the control system design data be overcome using high authority actuators or other control system hardware and software. Then the problem of high maneuver loads, low flutter speeds, static instability, etc. on a flight vehicle, that is designed to be aerodynamically optimum, can be minimized or removed through the use of a sophisticated CCV-type control system. Sone examples of the CCV-type control systems may be found in references 5, 6, and 7.

The achievements of benefits due to modern flight control concepts depends upon an accurate mathematical model of the aircraft. The mathematical model becomes even more important for CCV concepts. This is because the design of the CCV integrated control systems requires an accurate prediction of the rigid body and structural dynamics of the aircraft being stabilized along with an estimate of the possible errors that may be included in the dynamics. In general, errors in the dynamics arise from two sources:

1. The inappropriate formulation of the equations of motion.

2. The non-precise aerodynamic, structural, and inertial design data set.

It has been traditional to attribute most analytical errors in the flight control design of comtemporary elastic aircraft to the design data set and many discussions of these errors are contained in the literature. Thus, very few discussions of the reasons for the errors in equation formulation are to be found, even though the inappropriate formulation can negate the most accurate design data supplied by the engineering disciplines.

There are many formulations of the equations of motion of an elastic aircraft currently used in the aerospace industry. To date, most of the formulations have resulted from *a priori* assumptions or have been forced upon the flight control design due to expediency. These types of inappropriate formulations result in moderate to large-sized risk factors in the flight control design. The risk factors are manifested as the large gain and phase margins used as the flight control system design criteria to permit large adjustments in the control system gains during flight tests of the prototype. More complex integrated control systems for a highly elastic, high performance aircraft, such as one using CCV concepts, will not have the luxury of high gain and phase margins because of their complexity and their use of each aerodynamic control surface for many purposes.

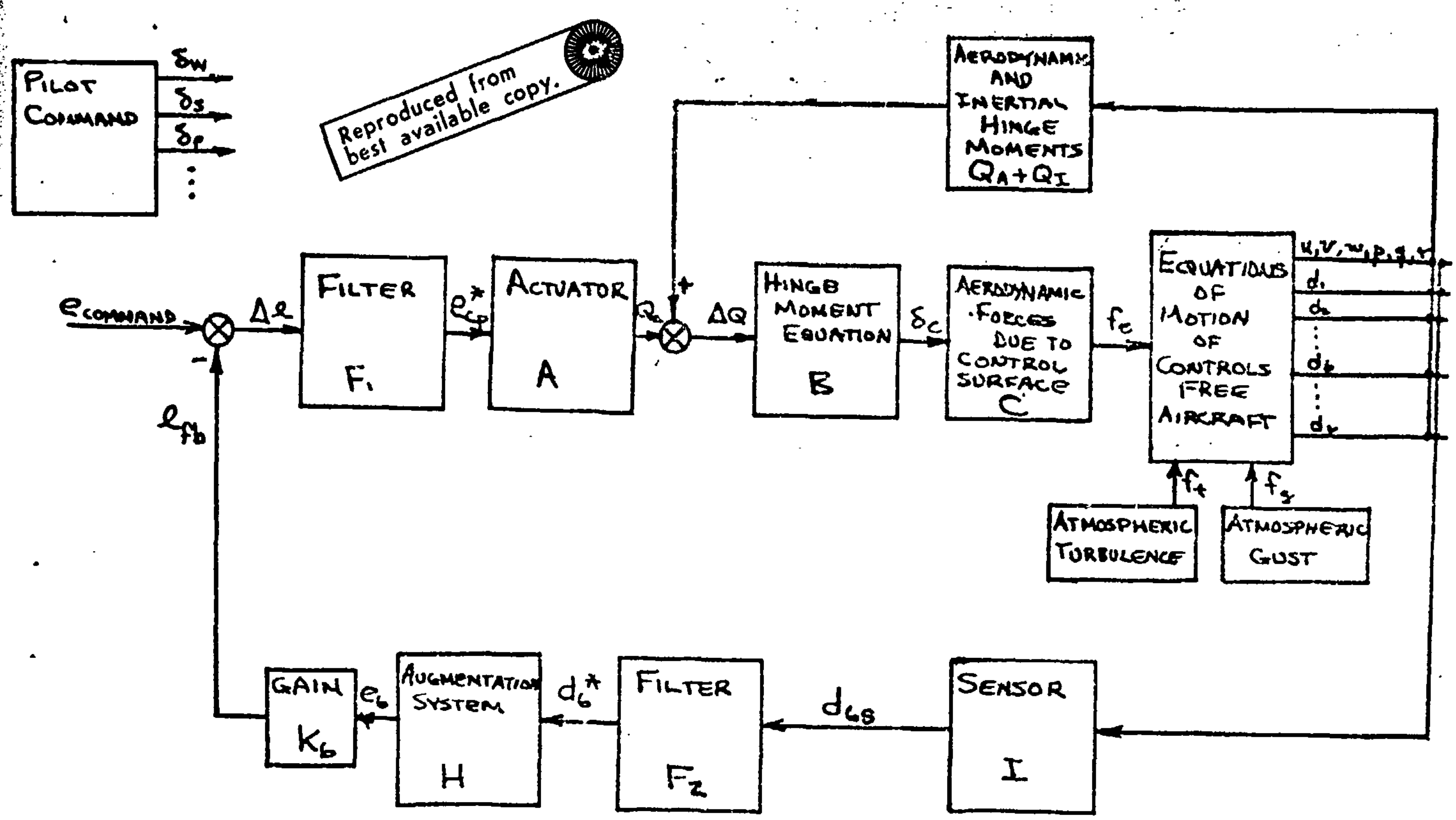
The effect of an inappropriate equation formulation on the analysis of the integrated flight control system can be visualized using an example. Figure 1.1, adapted from reference 8, portrays a simple feedback control system: the schematic is presented in Figure 1.1a and the stability matrix representation of the schematic in Figure 1.1b. In both cases, the equations of motion for the elastic airplane are shaded. As shown in the schematic, the equations of motion hold a dominant position in the design of integrated control systems. Their transfer function relates input quantities such as atmospheric turbulence and gusts and distributed control forces and moments to the output motion variables that determine the motion of the elastic aircraft through the atmosphere. The determination of the numerical value of all the elements in the schematic may be achieved by the evaluation of the characteristic equation of the determinant of the stability matrix in Figure 1.1b. As indicated, the equations of motion in the upper lefthand corner of the matrix influence the numerical value of all the control system elements, e.g., the filters, forward loop compensation, feedback loop compensation, sensors, sensor position, etc.

It is the intent of this paper to outline the source of those errors in the integrated control system due to the formulation of the equations of motion. This objective is accomplished by deriving a unique and precise set of linear, ordinary differential equations of motion for a controlled aircraft flying in a gusting or turbulent atmosphere (EXACT formulation). These precise equations are then simplified to five other formulations using identified assumptions. All formulations reported in the literature and used within the aerospace industry can be related to the EXACT formulation or to the five simplified formulations. A contrast of the mathematical model of each formulation to the actual physics of the elastic airplane determines:

- Which formulation is the most appropriate to each elastic airplane analysis.

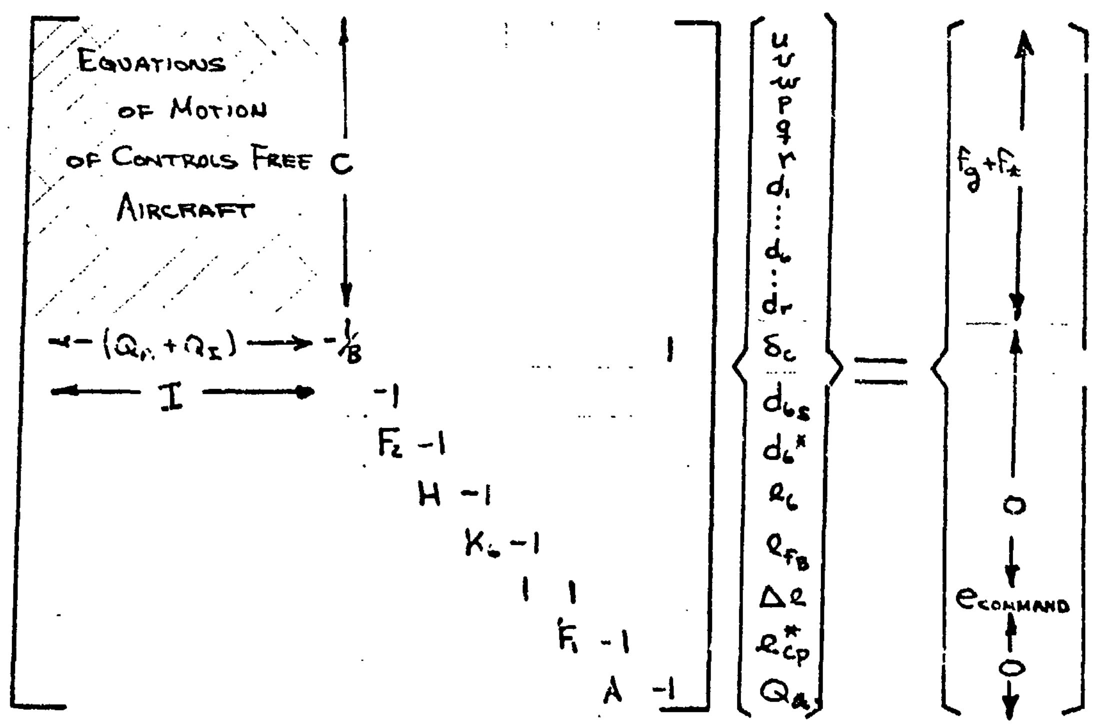
- What qualitative errors are included in the integrated control system design, if an inappropriate formulation is selected due to cost or time considerations.

In addition, an examination of each of the equation sets indicates:



a. SCHEMATIC OF A SIMPLE FEEDBACK CONTROL SYSTEM

$$[SM]\{X_o\} = \{X_I\}$$



b. STABILITY MATRIX OF A SIMPLE FEEDBACK CONTROL SYSTEM

FIGURE 1.1: IMPORTANCE OF EQUATIONS OF MOTION IN ANALYSIS OF INTEGRATED CONTROL SYSTEMS OF ELASTIC AIRCRAFT

- What problems are involved in implementing the formulations on limited size flight simulators.

- What formulation is particularly appropriate to the inclusion of nonlinear aerodynamics.

1.1 History and Terminology

The analysis of the dynamics of elastic aircraft is traceable to the flutter and control surface divergence aeroelastic problems encountered on the early military aircraft. As elastic aircraft became more complex, the analysis of these and other aeroelastic problems specialized to those engineering disciplines characterized as "flutter", "static structural loads", "dynamic structural loads", "configuration aerodynamic development", and "aeroelastic stability and control". With the advent of more modern aircraft such as the B-47, B-52, XB-70, YF-12, C5A, 747, SST and B-1, the interactions of these specialized disciplines of aeroelastic analysis became particularly pronounced during the design of the flight control system. The different terminology and equation formulations of each discipline result in engineering confusion and mis-matched structural, aerodynamic, and stability and control data.

It is the responsibility of the flight control engineer to integrate the data created by the aeroelastic disciplines to create a safe and useful aircraft by means of augmentation systems. These aircraft augmentation systems are used to improve:

- Handling qualities.

- Ride quality.

- Static stability.

- Fatigue life.

- Flutter margin.

- Maneuver loadability.

- Atmospheric gust loadability.

The integration of the diverse aeroelastic data is not usually performed by any of the other disciplines because of their important responsibilities to the detailed structural, aerodynamic, or stability and control design of the aircraft. An additional difficulty encountered by the other disciplines is that each of them has only a partial understanding of the terminology of each of the other disciplines, thus, the intercommunication required for integrated control system development is difficult[9].

Some preliminary formulations of the equations of motion of an elastic aircraft have been completed. Etkin[10] developed the equations of motion of rigid airplanes and outlined a method to be used to analyze elastic aircraft. More specific descriptions of Etkin's method and other methods are presented in references 11, 12, 13, and 14 along with illustrative applications to contemporary aircraft. There are some publications discussing the equations that describe the motions of complex elastic aircraft configurations. Unfortunately, these publications specialize the equations to an undamped, uncontrolled elastic aircraft[15] or to a particular formulation[16, 17, 18].

Those equation formulations commonly found in the literature may be categorized as:

QUASI STATIC - The motions of the structure are assumed to be in phase with the rigid body motions: elastic motion acceleration is instantaneous. The method is used primarily for handling quality and reduced static stability control system design for elastic aircraft with wide frequency separation between the rigid body and elastic motions.

EXACT - The motion of the structure is determined by the eigenvalue (root) and eigenvector (mode shape) solutions of the equations of motion for the elastic aircraft. The mode shape coordinates contain complex numbers. The accuracy of the solution is limited by the existing computerized routines that calculate the complex number eigenvalues and eigenvectors.

MODAL SUBSTITUTION - The motions of the structure are assumed to be related to the orthogonal, invacuum eigenvectors (mode shapes). All eigenvectors contain only real numbers.

RESIDUAL STIFFNESS - The mode shapes representing the elastic motion in the MODAL SUBSTITUTION formulation are separated into "retained" and "deleted" modes. The deleted modes are represented in the dynamic stability analysis as quasi static aeroelastic corrections, using a correction factor related to the deleted modes and the stiffness of the "free-free" structure.

RESIDUAL FLEXIBILITY - Similar to the RESIDUAL STIFFNESS formulation, except the quasi static aeroelastic correction factor is related to the retained modes and the flexibility of the free-free structure.

MODAL TRUNCATION - The deleted modes of the RESIDUAL FLEXIBILITY formulation are not represented by any correction factor. This is the most common dynamic aeroelastic formulation reported in the literature.

A unique qualitative/semiquantitative study of the QUASI STATIC RESIDUAL FLEXIBILITY, and MODAL TRUNCATION formulations was conducted by MacNeal, Schwendler, and Pearce for the USAF and reported in references 16, 18, and 19. These formulations were compared to the RIGID AIRPLANE and EXACT formulations using an elastic missile, a low aspect ratio aircraft, and a B-47 aircraft as study configurations. The aerodynamic and structural theory applied in this analysis was extremely "crude", but since the aerodynamic model was held constant for each configuration, its errors are hopefully minimized in the comparisons. Typical results of the study[16] for the elastic missile and the B-47 aircraft are presented in Figure 1.2, a Bode plot of the log amplitude of $\dot{\theta}/\delta_c$ versus frequency. The following conclusions were noted in the reference 16:

- At low frequencies (less than 0.50 cps in the Figure 1.2) the QUASI STATIC, RESIDUAL FLEXIBILITY, and MODAL TRUNCATION results are nearly coincident.

- At large frequencies (greater than 0.50 cps in the Figure 1.2) the QUASI STATIC formulation is highly inaccurate.

- Neither the RESIDUAL FLEXIBILITY nor the MODAL TRUNCATION formulations accurately approximate the EXACT formulation at all frequencies (above 0.50 cpc in the Figure 1.2). However, the RESIDUAL FLEXIBILITY formulation is the more accurate approximation.

The work presented in the reference 16 culminated the analytical efforts of the USAF to represent the motion of elastic aircraft. Most USAF analytical work in this field was discontinued in 1962 due to the absence of a digital computer system large enough to solve the complex equations developed using the structural and aerodynamic mathematical models that describe the physics of the elastic aircraft. Instead, the USAF concentrated on experimental programs such as the XB-70 GASDSAS and B-52 LAMS to demonstrate related concepts.

In 1965 - 1967 the analytical work of MacNeal, et. al. was studied by NASA and the Boeing Company and reported in reference 12. Here, it was recommended that the RESIDUAL FLEXIBILITY formulation be implemented using structural data from the structural finite element program such as NASTRAN, and using the Woodward aerodynamic finite element[20]. Also, the formulations of reference 16 were reworked to form a amenable to the large digital computer system - the CDC 6600 or IBM 360. The resulting program has been called FLEXSTAB[21]. User experience with the QUASI STATIC and RESIDUAL FLEXIBILITY formulations in FLEXSTAB has not yet been reported.

1.2 Problem Definition and Solution

The aerospace industry uses many of the formulations described in

FIGURE 1.2a

CONFIGURATION 2, MISSILE WITH TAIL AND CANARD

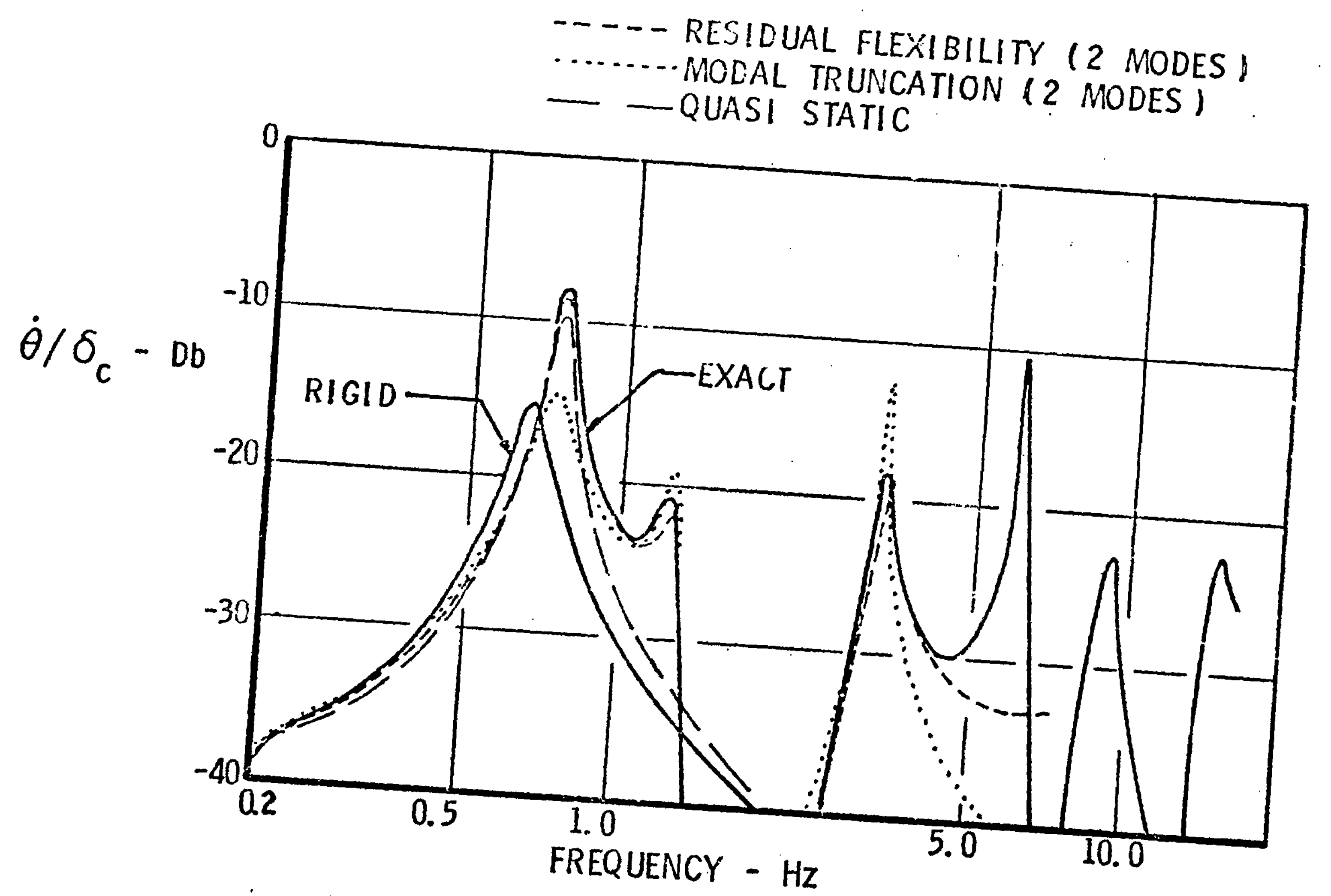
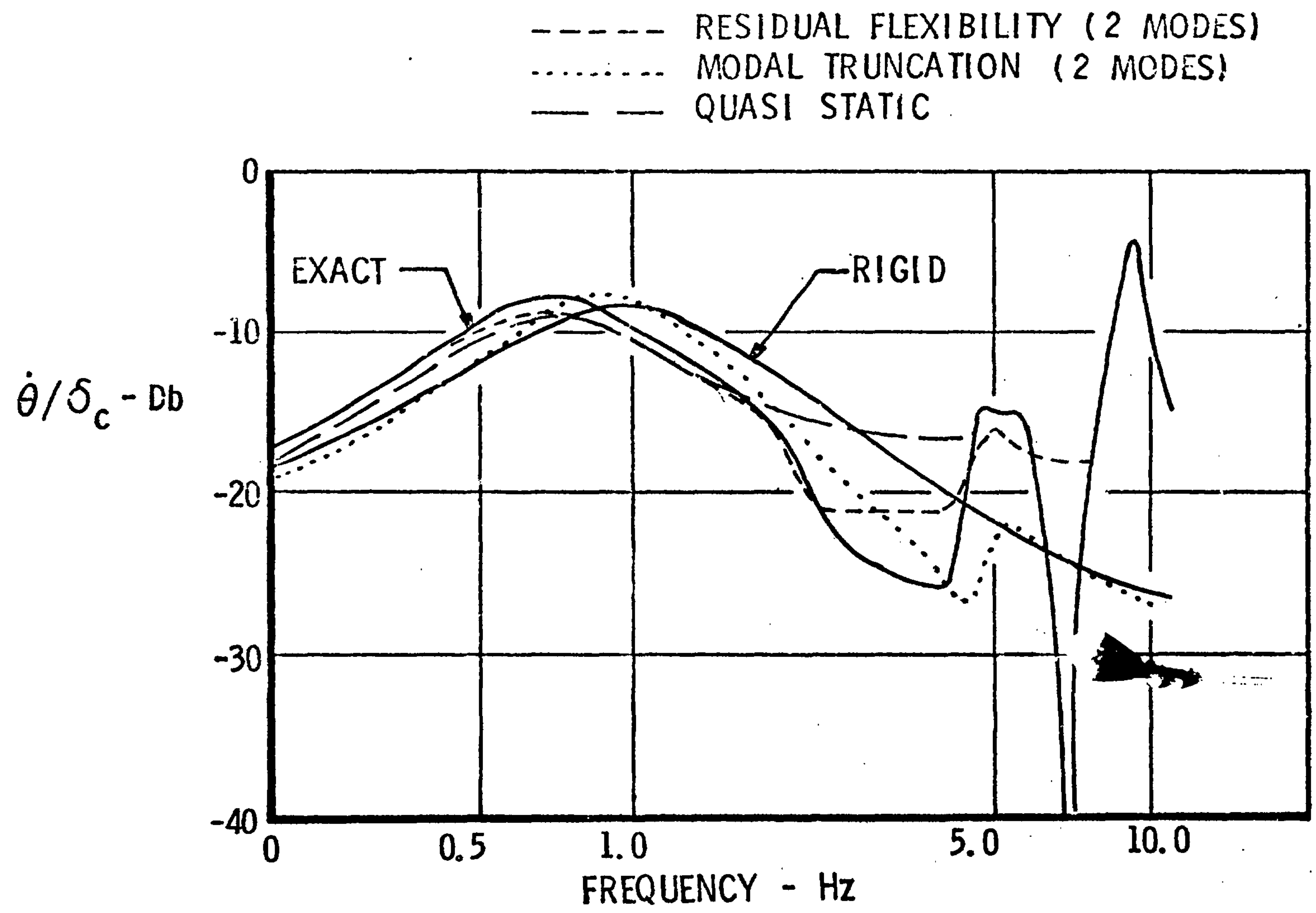


FIGURE 1.2b **CONFIGURATION 3, HIGH ASPECT RATIO, B-47 TYPE**



Section 1.1 to design and develop its military and commercial aircraft. Thus, it is common for the USAF and the AFFDL to encounter the many formulations (with individual company variations) during the proposal evaluation and the contract monitoring phases of military aircraft development. As examples:

The XB-70: QUASI STATIC and MODAL TRUNCATION.

The B-52-LAMS and B-52-CCV: MODAL TRUNCATION.

The F-4 CCV, Terrain Follower, and Survivable Flight Controls: QUASI STATIC and MODAL TRUNCATION.

The YF-12 RIDE QUALITY: QUASI STATIC and MODAL TRUNCATION.

The F-111: QUASI STATIC and MODAL TRUNCATION.

The C-5A: QUASI STATIC and MODAL TRUNCATION.

The B-1: QUASI STATIC and MODAL TRUNCATION.

The F-15: QUASI STATIC and MODAL TRUNCATION.

The proposed SST designs: QUASI STATIC, RESIDUAL STIFFNESS, and MODAL TRUNCATION.

There has been no previous attempt to categorize the formulations of the equations of motion used in flight control system designs and establish criteria for their use. As a consequence, the limitations of each formulation as dictated by the assumptions required to derive the equations are not recognized.

It is the objective of this Technical Memo to classify and contrast these formulations relative to a unique and precise formulation of the equations developed herein. This objective is accomplished in Sections 2.0 through 5.0. In Section 2.1 and 2.2 the nonlinear and linear equations are developed to describe the motion of a controls-fixed elastic aircraft relative to a non-inertial, body-fixed coordinate system. The linear equations are modified in Section 2.3 to include the effect of the free aerodynamic control surfaces and the effects of atmospheric gusts and turbulence on the elastic aircraft; the resulting equations are termed the EXACT formulation. In Section 3.0, the EXACT formulation is modified using listed assumptions to simplify the mathematical model of the elastic aircraft. These simplifications result in the formulations termed QUASI STATIC, MODAL SUBSTITUTION, RESIDUAL STIFFNESS, RESIDUAL FLEXIBILITY, and MODAL TRUNCATION. The results of the study are presented in Section 4.0; the conclusions and recommendations of the study are summarized in Section 5.0. A List of References is contained in Section 6.0.

It is not the intent of this paper to discuss the many methods available within the government and industry for the calculation of aerodynamic and structural influence coefficients. Thus, it is assumed throughout this paper that methods exist to calculate:

- Distributed aerodynamic forces on the surface of the aircraft.
- Mass distribution for the aircraft.
- Stiffness and flexibility matrices for the free-free and cantilevered structure.
- Other data as required for the flight control analyses.

Also the effect of thrust on the initial shape is assumed negligible. The forces due to thrust perturbations are included implicitly in the aerodynamic force terms.

2.0 DERIVATION OF THE EQUATIONS OF MOTION OF AN ELASTIC AIRCRAFT

Before the flight control engineer can begin development of the integrated flight control system of an elastic airplane, there must be an accurate description of the aircraft being controlled. This description is termed the "equations of motion" and the description contains implicit and explicit references to the structural, aerodynamic, and geometric properties of the aircraft. All the elastic aircraft equation of motion formulations have these three groups of data in common; however, the assembly of information depends upon the description of the dynamics of the aeroelastic problem.

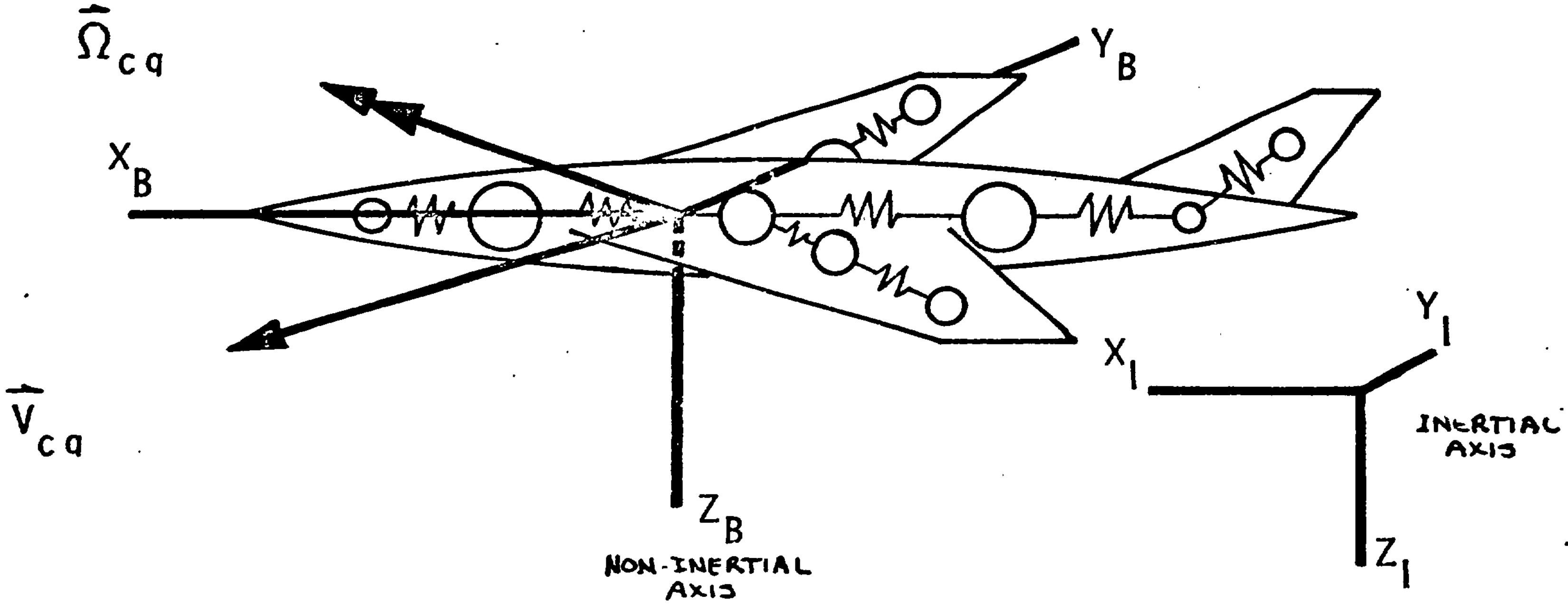
2.1 Selection of Motion Coordinates

There are many descriptions of the dynamics of elastic aircraft discussed in the literature. These descriptions can be grouped into two categories depending upon the coordinates used to describe the motions of the elastic aircraft. The two categories of displacement coordinates are:

- Displacement coordinates relative to an inertial axis, i.e., an earth fixed axis or any Galilean related axis.
- Displacement coordinates relative to a non-inertial axis, i.e., an axis "fixed" to the accelerating elastic aircraft.

These two axis systems are shown in Figure 2.1.

FIGURE 2.1 **LUMPED MASS IDEALIZATION OF AIRCRAFT**



2.1.1 Displacement relative to inertial coordinates

In the first analysis grouping, the displacement of the surface of the elastic aircraft is measured by a vector expressed relative to an inertial base vector system. Typical examples of this approach are found in references 11 and 17. A flight control engineer using this analysis approach to the dynamics requires that the structural, aerodynamic, and geometric vector parameters of the airplane be expressed in the inertial axis base vectors. It is usual in most of the literature to develop the elastic body motions as a trajectory of the body and the surface of the body relative to an inertial axis translating in a straight line at a uniform velocity relative to the flat earth. The uniform velocity and direction of motion is that of the elastic body prior to the onset of the disturbance producing the elastic motion. This analysis method is typically used by those specialities concerned with "flutter" and "dynamic structural loads".

2.1.2 Displacement relative to non-inertial coordinates

In the second analysis grouping the displacement of the surface of the elastic aircraft is measured by a vector expressed relative to a non-inertial axis base vector system. This axis system experiences both linear and angular accelerations relative to the inertial axis discussed previously. Typical examples of this approach are found in references 12 and 15. The flight control engineer utilizing this analysis approach requires that the structural, aerodynamic, and geometric vector properties of the aircraft be expressed in the body-fixed base vectors. This analysis approach is typically used by the specialities concerned with "aerodynamic configuration development" and "aeroelastic stability and control".

2.1.3 Contrast of inertial and non-inertial coordinates

The flight control engineer must choose the better of these two approaches for each elastic aircraft flight control integration problem. The inertial coordinate analysis is a classical mathematical approach and is familiar to most engineers who solve structural dynamics problems. On the other hand, the non-inertial coordinate analysis possesses considerable flight control engineering practicality:

1. Augmentation system criteria: The criteria presently used to design the handling quality and reduced static stability augmentation systems are expressed in terms of the rigid body motions and aerodynamic stability and control derivatives measured relative to the non-inertial axis. Since criteria for the integrated flight control augmentation system utilizing CCV concepts are not yet specified, either of the two analysis techniques would work. However, the non-inertial analysis technique has the prerogative.

2. Analysis of nonlinear aerodynamic effects: The integrated flight control system design points usually lie at the extremes of the flight placards where viscous aerodynamic effects induce pitch-up, buffet, etc. These effects are currently measured in the non-inertial, body-fixed axis coordinates. The important nonlinear aerodynamic and control system parameters are most efficiently programmed in terms of the non-inertial coordinates, thus minimizing the required computer size, computational frame time, and required axis transformations.

3. Flight simulator analysis: Pilot work load and misorientation are very important considerations for the design of an integrated flight control system. These two problems are usually evaluated in fixed base and moving base flight simulators. Since the pilot considers himself a "mass" attached to the airframe, he evaluates his motions relative to some body-fixed, non-inertial axis. Usually the information presented to him in the cockpit, i.e., angle of attack, sideslip angle, relative velocity, et., are directly related to the motion of the non-inertial axis.

4. Large disturbance maneuvers: Large disturbance military maneuvers required for defensive and offensive weapons delivery, are very often design points for an integrated control system. These large disturbance motions are easily described in terms of the non-inertial coordinates and are difficult to describe in terms of inertial coordinates.

Due to the above practical considerations, the non-inertial coordinate analysis approach is adopted in this Technical Memo. C viously, both analysis methods must give the same results, but the non-inertial analysis is the analysis method most easily applied to the flight control integration problem.

2.2 Controls-Fixed, Elastic Aircraft Equations of Motion

This section presents the equations that describe the motion of an elastic aircraft subject to aerodynamic forces that are independent of control surface position and atmospheric disturbances. The equations are derived for the case of the aircraft idealized as N "lumped masses" each having 3 translational degrees of freedom relative to a "body axis system". The 3N rotational degrees of freedom may easily be included in this formulation, but are omitted for purposes of clarity. The body axis system is defined as an axis system "attached" to the elastic aircraft by a means to be specified later in this development. Since the axis is attached, they are characterized as "non-inertial", i.e., the axis experiences 3 translational and 3 rotational accelerations whose magnitude are related to:

- The elastic motions of the N masses relative to the body axis.
- The aerodynamic forces on the system.
- The choice of the body axis.
- The location and orientation of the body axis at t = 0 (the initial condition).
- The gravitational forces acting on the aircraft.
- The mass distribution of the elastic aircraft.

The basis of this derivation is Lagrange's equation rather than the Newtonian equations applied by references 12 and 15. The reason for this approach is its simplicity of application - a minimum of notation and of analysis steps are required. The final results of the derivation can be checked against reference 12.

The Lagrangian technique to be followed is from reference 22 and that notation is used. Recall that the Lagrangian, L, must be written relative to an inertial axis to be used in Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{q}}\right) - \frac{\partial L_i}{\partial q} = \vec{F}_i \tag{2.1}$$

$$\vec{P}_i \times \frac{\partial L_i}{\partial \dot{q}} = \vec{h}_i \tag{2.2}$$

where L_i is the Lagrangian for some ith mass particle.

$\vec{q}$ is an appropriate coordinate vector.

$\partial L_i/\partial \vec{q}$ is the rate of change of L_i due to $\vec{q}$.

$\frac{d}{dt}(\sim)$ is the time rate of change of $(\sim)$ relative to an observer in inertial space.

$\vec{F}_i$ is the impressed force vector acting on the system.

$\vec{h}_i$ is the angular momentum vector of the system.

$\vec{P}_i$ is a distance from the axis origin to a mass point.

Lagrange's method can also be used to develop the equations of motion for an aircraft whose motions are measured relative to a non-inertial axis, i.e., base vectors $\hat{i}_B = \hat{i}_B(t)$, $\hat{j}_B = \hat{j}_B(t)$, and $\hat{k}_B = \hat{k}_B(t)$. The only additional condition to be applied to the analysis is an appropriate definition of $\frac{d}{dt}(\sim)$:

$$\frac{d}{dt}(\sim) = \frac{\delta}{\delta t}(\sim) + \vec{\omega} \times (\sim)$$

time rate of change relative to inertial axis	time rate of change relative to non-inertial axis	time rate of change of base vectors

Consider the case of the elastic aircraft, shown in Figure 2.1, which is experiencing an acceleration through a quiescent atmosphere at a translational velocity $\vec{V}_{ca}$ and a rotational rate $\vec{\Omega}_{ca}$. The aircraft should be visualized in terms of a system of "lumped masses" interconnected by "springs" and excited by external aerodynamic and gravitational forces.

It is important to realize that the body axis selection for the elastic aircraft case is very difficult:

- The geometric axis attached to selected masses points in the initial condition is non-orthogonal at subsequent time, i.e., the body reference axis system (BS, WBL, WL) becomes non-orthogonal for the free elastic aircraft.

- The principle axis system develops translational and rotational accelerations related to the center of mass motion plus the elastic motion.

- The attached axis system, attached to a single mass such as an accelerometer and tangent to the jig body axis at the initial condition, experiences acceleration proportional to the center of mass motion plus the motion of the individual mass relative to the center of mass motion.

There are several constraints to the choice of the body axis system:

- The aerodynamic data is usually available only in the jig or cruise geometric axis system, the stability axis system, or the wind axis system.
- The mass distribution and geometry are specified in the jig or cruise geometric axis system.
- The structural properties, stiffness and flexibility, are specified in a structural global axis system that may be coincident with the inertial axis or the non-inertial axis at the initial conditions.

It is assumed for purposes of this derivation of the equations of motion of the elastic aircraft, that the mass distribution, aircraft geometry, and structural data are specified in the cruise body axis system. Then, the methods of reference 21 or 23 may be applied to transform these data to a representation of the aircraft at any trimmed, off-cruise, M-q flight point. It is also assumed that subsequent structural distortions from the trimmed aircraft flight point are small, i.e., changes in the geometry of the planform and the effects of material plasticity can be neglected. The primary structural distortions considered are those contributing significant forces during some perturbation motion about the mean motion specified by the initial conditions of the dynamics problem.

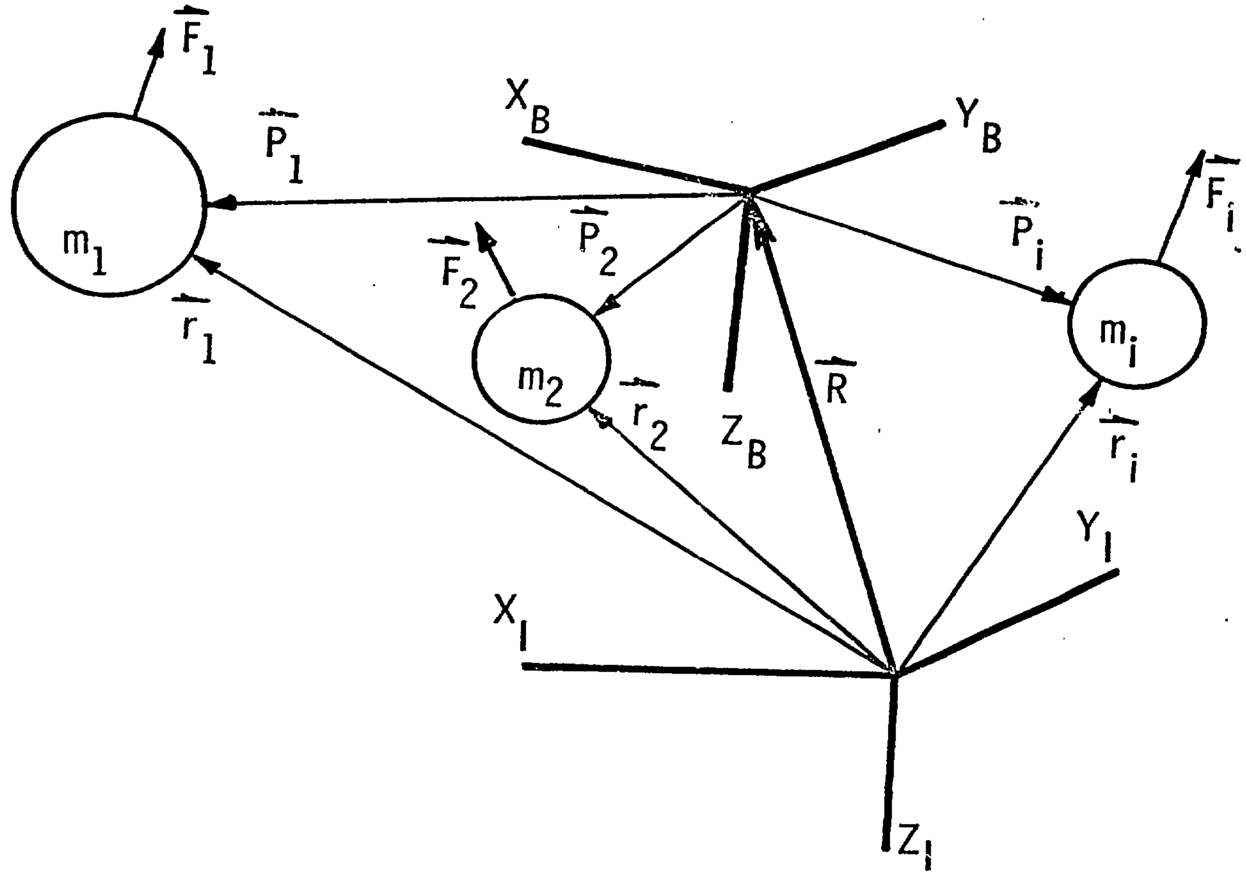
A more microscopic view of Figure 2.1 is useful in defining the terms used in the derivation of the elastic airplane equations of motion. One such view is presented in Figure 2.2. Here the inertial axis, the body axis, and several of the lumped masses, including the ith mass are shown. Each of these lumped masses represents a portion of the total airplane mass, i.e., fuel, payload, structure, instruments, etc. Each of the masses has gravitational and aerodynamic forces acting upon it. Each of the masses has accelerations imposed upon it proportional to the net system accelerations (body axis accelerations).

The notation used in Figure 2.2 is defined:

$X_B Y_B Z_B$ are non-inertial axis attached to the elastic aircraft; base vectors $\hat{i}_B$, $\hat{j}_B$ and $\hat{k}_B$.

$X_I Y_I Z_I$ are the inertial axis either attached to the flat earth or translating rectilinearly at a uniform velocity; base vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

FIGURE 2.2 **MICROSCOPIC VIEW OF i th MASS**



$\vec{R}(t)$ is a vector distance from the inertial axis origin to the body axis origin.

$\vec{p}_i(t)$ is a vector distance from the body axis origin to the ith mass.

$\vec{r}_i(t)$ is the vector distance from the inertial axis origin to the ith mass.

$\vec{F}_i(t)$ is the vector force on the ith mass.

The Lagrangian for any mass in this system is:

$$L_i = 1/2\, m_i (\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i) - 1/2 \sum_{j=1}^{N} K_{ij} (\vec{p}_i \cdot \vec{p}_j) \tag{2.3}$$

where m_i is the mass of the ith "lumped mass".
K_{ij} is the free-free structural stiffness influence coefficient at the ith mass due to a displacement at the jth mass in one of three possible directions.

Applying the definition of $\vec{r}_i$, shown in Figure 2.2,

$$\dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{p}}_i = \vec{V}_{ca} + \frac{\delta \vec{p}_i}{\delta t} + \vec{\Omega}_{ca} \times \vec{p}_i \tag{2.4}$$

$$L_i = 1/2\, m_i \left((\dot{\vec{R}} + \dot{\vec{p}}_i) \cdot (\dot{\vec{R}} + \dot{\vec{p}}_i) \right) - \frac{1}{2} \sum_{j=1}^{N} K_{ij} (\vec{p}_i \cdot \vec{p}_j) \tag{2.5}$$

Equation (2.5) contains 6+3 N motion variables.

Substitution of L_i from equation (2.5) into equation (2.1), and letting $\vec{q} = \vec{p}_i$, results in the equations of motion for the 3 translational degrees of freedom of the ith mass relative to the non-inertial axis:

$$\begin{aligned} \vec{F}_i = m_i \Big(\dot{\vec{V}}_{ca} + \vec{\Omega}_{ca} \times \vec{V}_{ca} + \frac{\delta^2 \vec{p}_i}{\delta t^2} + 2 \vec{\Omega}_{ca} \times \frac{\delta \vec{p}_i}{\delta t} \\ + \dot{\vec{\Omega}}_{ca} \times \vec{p}_i + \vec{\Omega}_{ca} \times (\vec{\Omega}_{ca} \times \vec{p}_i) \Big) \\ + \sum_{j=1}^{N} K_{ij} \vec{p}_j \end{aligned} \tag{2.6a}$$

For axis translation, $\vec{q} = \vec{R}$, $\dot{\vec{q}} = \dot{\vec{R}}$, $L = \sum_{i=1}^{N} L_i$:

$$\partial L/\partial \vec{R} = 0 \quad , \quad \partial L/\partial \dot{\vec{R}} = \sum_{i=1}^{N} m_i \left(\vec{V}_{ca} + \delta\vec{\rho}_i/\delta t + \vec{\Omega}_{ca} \times \vec{\rho}_i\right)$$

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} m_i \Big(\dot{\vec{V}}_{ca} + \vec{\Omega}_{ca} \times \vec{V}_{ca} + \delta^2\vec{\rho}_i/\delta t^2 + 2\vec{\Omega}_{ca} \times \delta\vec{\rho}_i/\delta t + \dot{\vec{\Omega}}_{ca} \times \vec{\rho}_i + \vec{\Omega}_{ca} \times (\vec{\Omega}_{ca} \times \vec{\rho}_i)\Big) \tag{2.6b}$$

The equations describing the axis rotation are more difficult to develop since the axis origin is not attached to the center of mass. This necessitates a careful consideration of the mathematical operation employed in equation (2.2). Consider first the angular momentum for the mass system in Figures 2.1 and 2.2 for observers stationed in both the inertial axis system with origin O_I, and in the non-inertial axis system with origin O_P :

$$\vec{h}_I = \sum_{i=1}^{N} \vec{r}_i \times m_i \dot{\vec{r}}_i \quad , \quad \vec{h}_P = \sum_{i=1}^{N} \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

Following the work of Goodman and Warner(24), the relationship of $\vec{h}_I$ to $\vec{h}_P$ can be developed:

$$\vec{h}_P = \vec{h}_I - \vec{\rho}^* \times M\dot{\vec{R}} - \vec{R} \times M\dot{\vec{r}}^*$$

where $\vec{\rho}^*$ is the distance from O_P to the center of mass.
M is the total mass, i.e., $M = \sum_{i=1}^{N} m_i$.

$\dot{\vec{r}}^*$ is the velocity of the center of mass relative to the inertial axis.

Differentitate $\vec{h}_P$, i.e., $d\vec{h}_P/dt$:

$$\dot{\vec{h}}_P = \underbrace{\vec{M}_I - \vec{R} \times M\vec{a}^*}_{} - M\vec{\rho}^* \times \ddot{\vec{R}}$$

$$= \vec{M}_P - M\rho^* \times \ddot{\vec{R}}$$

where $\vec{M}_I$ are the moments about O_I .

$\vec{M}_P$ are the moments about O_P .

$\vec{a}^*$ is the acceleration of the center of mass relative to the inertial axis.

Thus, $d\vec{h}_p/dt \neq \vec{M}_p$ unless the non-inertial axis is attached to the center of mass, $\vec{p}^* = 0$, or $\vec{R} = 0$, or $\vec{p}^*$ is parallel to $\vec{R}$. An alternate form for $\vec{h}_p$ is that presented by Milne[15]. Here,

$$\vec{h}_p = \sum_{i=1}^{N} \vec{p}_i \times m_i\, d\vec{r}_i/dt$$

$$\dot{\vec{h}}_p = \vec{M}_p - \vec{V}_{ca} \times \sum_{i=1}^{N} m_i\, d\vec{p}_i/dt$$

A convenient redefinition of this expression for $\dot{\vec{h}}_p$ separates the aerodynamic and thrust moments $\vec{M}_a$, from the gravitational moments, $\vec{M}_G$, i.e.,

$$\dot{\vec{h}}_p = \vec{M}_a - \vec{V}_{ca} \times \sum_{i=1}^{N} m_i\, d\vec{p}_i/dt + \vec{M}_G$$

or, alternatively,

$$\vec{M}_a = \dot{\vec{h}}_p + \vec{M}_g$$

where $\vec{M}_g = \vec{V}_{ca} \times \sum_{i=1}^{N} m_i\, \delta\vec{p}_i/\delta t + \vec{V}_{ca} \times \sum_{i=1}^{N} m_i (\vec{\Omega}_{ca} \times \vec{p}_i) - \vec{M}_G$

for an axis system not attached to the center of mass.

$= 0$ for the axis attached to the center of mass.

In terms of the notation in this paper, $\vec{h}_p$ and $\vec{M}_a$ are expressed as:

$$\vec{h}_p = \sum_{i=1}^{N} \vec{p}_i \times m_i (\vec{V}_{ca} + \delta\vec{p}_i/\delta t + \vec{\Omega}_{ca} \times \vec{p}_i)$$

$$= \sum_{i=1}^{N} \vec{p}_i \times m_i \vec{V}_{ca} + \sum_{i=1}^{N} \vec{p}_i \times m_i \delta\vec{p}_i/\delta t + \sum_{i=1}^{N} \vec{p}_i \times m_i (\vec{\Omega}_{ca} \times \vec{p}_i)$$

$$\begin{aligned}
\vec{M}_a = {} & \delta\vec{h}_p/\delta t + \vec{\Omega}_{ca} \times \vec{h}_p + \vec{M}_g \\
= {} & \sum_{i=1}^{N} m_i (\delta\vec{p}_i/\delta t \times \vec{V}_{ca}) + \sum_{i=1}^{N} m_i (\vec{p}_i \times \dot{\vec{V}}_{ca}) + \sum_{i=1}^{N} m_i \vec{\Omega}_{ca} \times (\vec{p}_i \times \vec{V}_{ca}) \\
& + \sum_{i=1}^{N} m_i \vec{p}_i \times (\dot{\vec{\Omega}}_{ca} \times \vec{p}_i) - \sum_{i=1}^{N} m_i \vec{\Omega}_{ca} \times \vec{p}_i (\vec{\Omega}_{ca} \cdot \vec{p}_i) \\
& + \sum_{i=1}^{N} m_i (\vec{p}_i \times \delta^2\vec{p}_i/\delta t^2) \\
& - 2\sum_{i=1}^{N} m_i \vec{p}_i \times (\delta\vec{p}_i/\delta t \times \vec{\Omega}_{ca}) + \vec{M}_g
\end{aligned} \tag{2.6c}$$

The reader should observe, at this point, that a simplification of equation (2.6c) is possible by noting that the terms $\sum m_i(\delta \vec{P}_i/\delta t \times \vec{V}_{ca})$ and the term $\vec{V}_{ca} \times \sum m_i \delta \vec{P}_i/\delta t$ will cancel one another. However, this cancellation will not be effected to measure the importance of the attachment of the axis orgin to the center of mass to occur later in this paper in section 2.2.2.

2.2.1 Large disturbance equations of motion

Equations (2.6) are the equations describing the general motion of elastic flight vehicles in the quiescent atmosphere. In practice, these equations are modified and simplified to facilitate their use in the analysis and synthesis of flight control systems. The description of the large disturbance motions, to be defined the "large disturbance equations", is required for two purposes in these analysis. They are used (1) to develop the linear equations describing the small disturbance motions and (2) to evaluate in detail the control systems designed using the more simple and less accurate linear equations.

To develop the large disturbance equations, it is assumed that all the time dependent motion variables can be separated into an initial value, plus a perturbation variable. In the case of equations (2.6a), (2.6b), and (2.6c), this separation has the form:

$$\vec{V}_{ca} = (V_j + v_j)\hat{i}_j = (U_1+u_1)\hat{i}_B + (U_2+u_2)\hat{j}_B + (U_3+u_3)\hat{k}_B \tag{2.7a}$$

$$\vec{\Omega}_{ca} = (\Omega_j + \omega_j)\hat{i}_j = (\Omega_1+\omega_1)\hat{i}_B + (\Omega_2+\omega_2)\hat{j}_B + (\Omega_3+\omega_3)\hat{k}_B \tag{2.7b}$$

$$\vec{P}_i = (P_{ji} + p_{ji})\hat{i}_j = (P_{1i}+p_{1i})\hat{i}_B + (P_{2i}+p_{2i})\hat{j}_B + (P_{3i}+p_{3i})\hat{k}_B \tag{2.7c}$$

A substitution of the perturbation expressions defined in equations (2.7), into the motions, related by equations (2.6), results in equations (2.8). In equations (2.8) vector products, i.e., $\vec{a} \times \vec{b}$, have been converted to an equivalent indicial notation form, i.e., $e_{jkl}a_k b_l$.

<u>Axis Translation:</u>

$$\begin{aligned}(F_j+f_j) = {} & M\left[(\dot{V}_j+\dot{v}_j) + e_{jkl}(\Omega_k+\omega_k)(V_l+v_l)\right] + m_i(\ddot{P}_{ji}+\ddot{p}_{ji}) \\ & + 2m_i e_{jkl}(\Omega_k+\omega_k)(\dot{P}_{li}+\dot{p}_{li}) + m_i e_{jkl}(\dot{\Omega}_k+\dot{\omega}_k)(P_{li}+p_{li}) \\ & + m_i e_{jkl}(\Omega_k+\omega_k)\, e_{lmn}(\Omega_m P_{ni} + \Omega_m p_{ni} + \omega_m P_{ni} + \omega_m p_{ni})\end{aligned}$$

where i ranges from 1 to N; where j, k, l, m, n range from 1 to 3.

(2.8a)

The term e_{jkl} is the vector permutation symbol commonly employed in the indicial notation to represent vector cross products.

Axis Rotation:

$$\left(M'_{a_j} + M_{a_j}\right) = \left(\dot{H}_{p_j} + \dot{h}_{p_j}\right) + \left(M'_{g_j} + M_{g_j}\right) \qquad (2.8b)$$

where

$$\begin{aligned}\left(\dot{H}_{p_j} + \dot{h}_{p_j}\right) = {} & m_i e_{jkl} (\dot{P}_{ki} + \dot{p}_{ki})(V_l + v_l) + m_i e_{jkl} (P_{ki} + p_{ki})(\dot{V}_l + \dot{v}_l) \\ & + m_i e_{jkl} (\Omega_k + \omega_k)\, e_{lmn} (P_{mi} + p_{mi})(V_n + v_n) \\ & + m_i e_{jkl} (P_{ki} + p_{ki})\, e_{lmn} (\dot{\Omega}_m + \dot{\omega}_m)(P_{ni} + p_{ni}) \\ & - m_i e_{jkl} (\Omega_k + \omega_k)(P_{li} + p_{li})\, \delta_{mn} (\Omega_m + \omega_m)(P_{ni} + p_{ni}) \\ & + m_i e_{jkl} (P_{ki} + p_{ki})(\ddot{P}_{li} + \ddot{p}_{li}) \\ & - 2 m_i e_{jkl} (P_{ki} + p_{ki})\, e_{lmn} (\dot{P}_{mi} + \dot{p}_{mi})(\Omega_n + \omega_n)\end{aligned}$$

$$\begin{aligned}\left(M'_{g_j} + M_{g_j}\right) = {} & m_i e_{jkl} (V_k + v_k)(\dot{P}_{li} + \dot{p}_{li}) \\ & + m_i e_{jkl} (V_k + v_k)\, e_{lmn} (\Omega_m + \omega_m)(P_{ni} + p_{ni}) \\ & - \left(M'_{a_j} + M_{a_j}\right)\end{aligned}$$

where δ_{mn} is the Kronecker delta function, where i ranges from 1 to N; where j, k, l, m, n ranges over 1, 2, and 3.

Elastic Deformation at ith Mass:

$$(F_{ji}+f_{ji}) = m_i\Big[(\dot{V}_j+\dot{v}_j)+e_{jkl}(\Omega_k+\omega_k)(V_l+v_l)+(\ddot{P}_{ji}+\ddot{p}_{ji})$$
$$+2e_{jkl}(\Omega_k+\omega_k)(\dot{P}_{li}+\dot{p}_{li})+e_{jkl}(\dot{\Omega}_k+\dot{\omega}_k)(P_{li}+p_{li})$$
$$+e_{jkl}(\Omega_k+\omega_k)e_{lmn}(\Omega_m P_{ni}+\Omega_m p_{ni}+\omega_m P_{ni}+\omega_m p_{ni})\Big]$$
$$+\sum_{k=1}^{N}\sum_{l=1}^{3}K_{ijkl}(p_{lk}+P_{lk}) \tag{2.8c}$$

where $K_{ijkl}p_{lk}$ is the force on the ith mass in the jth direction due to a deformation at the kth mass in the lth direction, and where no summation on i is intended; where j, k, l, m, n range over 1, 2, and 3 unless otherwise noted.

In equations (2.8), the initial values of the parameters are F_j, $\dot{H}_{p_j}$, M'_{a_j}, M'_{g_j}, M'_{G_j}, F_{ji}, P_{ji}, V_j, and Ω_j; the perturbation values of the parameters are f_j, h_{p_j}, M_{a_j}, M_{g_j}, M_{G_j}, f_{ji}, p_{ji}, v_j and ω_j.

Equations (2.8) contain both initial condition motions and perturbation motions from the initial conditions. It is of advantage in later analysis to now separate the two motions. This can be accomplished by setting all the perturbation parameters to zero in equations (2.8):

Axis Translation:

$$F_j = M[\dot{V}_j+e_{jkl}\Omega_k V_l]+m_i\ddot{P}_{ji}+2m_i e_{jkl}\Omega_k\dot{P}_{li}$$
$$+m_i e_{jkl}\dot{\Omega}_k P_{li}+m_i e_{jkl}\Omega_k e_{lmn}\Omega_m P_{ni} \tag{2.9a}$$

Axis Rotation:

$$M'_{a_j} = \dot{H}_{p_j}+M'_{g_j} \tag{2.9b}$$

where

$$\dot{H}_{p_j} = m_i e_{jkl}\dot{P}_{ki}V_l+m_i e_{jkl}P_{ki}\dot{V}_l+m_i e_{jkl}\Omega_k e_{lmn}P_{mi}V_n$$
$$+m_i e_{jkl}P_{ki}e_{lmn}\dot{\Omega}_m P_{ni}-m_i e_{jkl}\Omega_k P_{li}\delta_{mn}\Omega_m P_{ni}$$
$$+m_i e_{jkl}P_{ki}\ddot{P}_{li}-2m_i e_{jkl}P_{ki}e_{lmn}\dot{P}_{mi}\Omega_n$$

$$M'_{g_j} = m_i e_{jkl} V_k \dot{P}_{li} + m_i e_{jkl} V_k e_{lmn} \Omega_m P_{ni} - M'_{G_j}$$

Elastic Deformation at ith Mass:

$$\begin{aligned} F_{ji} = m_i [& \dot{V}_j + e_{jkl} \Omega_k V_l + \ddot{P}_{ji} + 2 e_{jkl} \Omega_k \dot{P}_{li} \\ & + e_{jkl} \dot{\Omega}_k P_{li} + e_{jkl} \Omega_k e_{lmn} \Omega_m P_{ni}] \\ & + \sum_{k=1}^{N} \sum_{l=1}^{3} K_{ijkl} P_{lk} \end{aligned} \qquad (2.9c)$$

A substraction of the initial conditions in equations (2.9) from the combined initial conditions and perturbation motions in equations (2.8) results in the large disturbance equations of motion, equations (2.10):

Axis Translation:

$$\begin{aligned} f_j = & M [\dot{v}_j + e_{jkl} (\Omega_k v_l + V_l \omega_k + v_l \omega_k)] + m_i \ddot{p}_{ji} \\ & + 2 m_i e_{jkl} (\Omega_k \dot{p}_{li} + \omega_k \dot{P}_{li} + \omega_k \dot{p}_{li}) \\ & + m_i e_{jkl} (\dot{\Omega}_k p_{li} + \dot{\omega}_k P_{li} + \dot{\omega}_k p_{li}) \\ & + m_i e_{jkl} (\Omega_k + \omega_k) e_{lmn} (\Omega_m p_{ni} + \omega_m P_{ni} + p_{ni} \omega_m) \\ & + m_i e_{jkl} \omega_k e_{lmn} \Omega_m P_{ni} \end{aligned} \qquad (2.10a)$$

Axis Rotation:

$$M_{a_j} = \dot{h}_{p_j} + M_{g_j} \qquad (2.10b)$$

where

$$
\begin{aligned}
\dot{h}_{p_j} = {} & m_i e_{jkl}(\dot{P}_{ki} v_l + \dot{p}_{ki} V_l + v_l \dot{p}_{ki}) + m_i e_{jkl}(P_{ki} \dot{v}_l + p_{ki} \dot{V}_l + p_{ki} \dot{v}_l) \\
& + m_i e_{jkl} \omega_k e_{lmn} P_{mi} V_n + m_i e_{jkl} (\Omega_k + \omega_k) e_{lmn} (P_{mi} v_n + p_{mi} V_n + p_{mi} v_n) \\
& + m_i e_{jkl} p_{ki} e_{lmn} \dot{\Omega}_m P_{ni} + m_i e_{jkl} (P_{ki} + p_{ki}) e_{lmn} (\dot{\Omega}_m p_{ni} + \dot{\omega}_m P_{ni} + \dot{\omega}_m p_{ni}) \\
& - m_i e_{jkl} \Omega_k P_{li} \delta_{mn} (\Omega_m p_{ni} + \omega_m P_{ni} + \omega_m p_{ni}) \\
& - m_i e_{jkl} (\Omega_k p_{li} + \omega_k P_{li} + \omega_k p_{li}) \delta_{mn} (\Omega_m P_{ni} + \Omega_m p_{ni} + \omega_m P_{ni} + \omega_m p_{ni}) \\
& + m_i e_{jkl} (P_{ki} \ddot{p}_{li} + p_{ki} \ddot{P}_{li} + p_{ki} \ddot{p}_{li}) \\
& - 2 m_i e_{jkl} p_{ki} e_{lmn} \dot{P}_{mi} \Omega_n \\
& - 2 m_i e_{jkl} (P_{ki} + p_{ki}) e_{lmn} (\dot{P}_{mi} \omega_n + \dot{p}_{mi} \Omega_n + \dot{p}_{mi} \omega_n)
\end{aligned}
$$

$$
\begin{aligned}
M_{g_j} = {} & m_i e_{jkl} (V_k \dot{p}_{li} + v_k \dot{P}_{li} + v_k \dot{p}_{li}) \\
& + m_i e_{jkl} v_k e_{lmn} \Omega_m P_{ni} \\
& + m_i e_{jkl} (V_k + v_k) e_{lmn} (\Omega_m p_{ni} + \omega_m P_{ni} + \omega_m p_{ni}) \\
& - M_{G_j}
\end{aligned}
$$

Elastic Deformation at ith Mass:

$$
\begin{aligned}
f_{ji} = m_i \Big[& \dot{v}_j + e_{jkl}(\Omega_k v_l + V_l \omega_k + \omega_k v_l) + \ddot{p}_{ji} \\
& + 2 e_{jkl}(\Omega_k \dot{p}_{li} + \omega_k \dot{P}_{li} + \omega_k \dot{p}_{li}) \\
& + e_{jkl}(\dot{\Omega}_k p_{li} + \dot{\omega}_k P_{li} + \dot{\omega}_k p_{li}) + e_{jkl}(\Omega_k + \omega_k) e_{lmn}(\Omega_m p_{ni} + \omega_m p_{ni} + \omega_m P_{ni}) \\
& + e_{jkl}\omega_k e_{lmn}\Omega_m P_{ni} \Big] + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{ijkl}\, p_{lk}
\end{aligned}
\qquad (2.10c)
$$

The large disturbance perturbation equations contain reference to both the initial conditions and to the perturbation motions. Usually, these initial conditions are specified or determined from equations (2.9) in each problem of interest to the flight control engineer.

2.2.2 Small distrubance equations of motion

In many cases of practical importance, the large disturbance equations are simplified to a small disturbance form. This simplification accomplished by assuming that all perturbation variables are of order ε ,$O(\varepsilon)$, where ε is a very small number. Then, products of the perturbation variables are of even smaller order, i.e., $O(\varepsilon)$ multiplied by $O(\varepsilon)$ is $O(\varepsilon^2)$, where $O(\varepsilon^2) \ll O(\varepsilon)$. If all terms of $O(\varepsilon^2)$ or smaller are eliminated, equations (2.10) are linearized, and the small disturbance equations result:

Axis Translation:

$$
\begin{aligned}
f_j = M\left[\dot{v}_j + e_{jkl}(\Omega_k v_l + \omega_k V_l)\right] & + m_i \ddot{p}_{ji} + 2 m_i e_{jkl}(\Omega_k \dot{p}_{li} + \omega_k \dot{P}_{li}) \\
& + m_i e_{jkl}(\dot{\Omega}_k p_{li} + \dot{\omega}_k P_{li}) + m_i e_{jkl}\Omega_k e_{lmn}(\Omega_m p_{ni} + \omega_m P_{ni}) \\
& + m_i e_{jkl}\omega_k e_{lmn}\Omega_m P_{ni}
\end{aligned}
\qquad (2.11a)
$$

Axis Rotation:

$$M_{a_j} = \dot{h}_{p_j} + M_{g_j} \tag{2.11b}$$

where

$$\dot{h}_{p_j} = m_i e_{jkl}(\dot{P}_{k_i} v_l + \dot{\rho}_{k_i} V_l) + m_i e_{jkl}(P_{l_i}\dot{v}_l + \rho_{l_i}\dot{V}_l)$$
$$+ m_i e_{jkl}\omega_k e_{lmn} P_{m_i} V_n + m_i e_{jkl}\Omega_k e_{lmn}(P_{m_i} v_n + \rho_{n_i} V_n)$$
$$+ m_i e_{jkl}\rho_{k_i} e_{lmn}\dot{\Omega}_m P_{n_i} + m_i e_{jkl} P_{k_i} e_{lmn}(\dot{\Omega}_m \rho_{n_i} + \dot{\omega}_m P_{n_i})$$
$$- m_i e_{jkl}\Omega_k P_{l_i}\delta_{mn}(\Omega_m \rho_{n_i} + \omega_m P_{n_i})$$
$$- m_i e_{jkl}(\Omega_k \rho_{l_i} + \omega_k P_{l_i})\delta_{mn}\Omega_m P_{n_i} + m_i e_{jkl}(P_{k_i}\ddot{\rho}_{l_i} + \rho_{k_i}\ddot{P}_{l_i})$$
$$- 2 m_i e_{jkl}\rho_{k_i} e_{lmn}\dot{P}_{m_i}\Omega_n$$
$$- 2 m_i e_{jkl} P_{k_i} e_{lmn}(\dot{P}_{m_i}\omega_n + \dot{\rho}_{m_i}\Omega_n)$$

$$M_{g_j} = m_i e_{jkl}(V_k \dot{\rho}_{l_i} + v_k \dot{P}_{l_i}) + m_i e_{jkl} v_k e_{lmn}\Omega_m P_{n_i}$$
$$+ m_i e_{jkl} V_k e_{lmn}(\Omega_m \rho_{n_i} + \omega_m P_{n_i})$$
$$- M_{G_j}$$

Elastic Deformation at ith Mass:

$$f_{j_i} = m_i\big[\dot{v}_j + e_{jkl}(\Omega_k v_l + \omega_k V_l) + \ddot{P}_{j_i} + 2 e_{jkl}(\Omega_k \dot{\rho}_{l_i} + \omega_k \dot{P}_{l_i})$$
$$+ e_{jkl}(\dot{\Omega}_k \rho_{l_i} + \dot{\omega}_k P_{l_i}) + e_{jkl}\Omega_k e_{lmn}(\Omega_m \rho_{n_i} + \omega_m P_{n_i})$$
$$+ e_{jkl}\omega_k e_{lmn}\Omega_m P_{n_i}\big]$$
$$+ \sum_{k=1}^{N}\sum_{l=1}^{3} K_{ijkl} P_{lk} \tag{2.11c}$$

The most common flight control problem analyzed using the small disturbance equations is that of wings level, constant velocity, rectilinear flight, parallel to the surface of the "flat" earth. As a consequence, the remainder of this paper will consider that analysis problem. For these initial conditions, equations (2.11) are greatly simplified:

Axis Translation:

$$f_j = M\left[\dot{v}_j + e_{jkl} V_l \omega_k\right] + m_i \ddot{p}_{ji} + m_i e_{jkl} \dot{\omega}_k P_{li} \quad (2.12a)$$

Axis Rotation:

$$M_{a_j} = \dot{h}_{p_j} + M_{g_j} \quad (2.12b)$$

$$\dot{h}_{p_j} = m_i e_{jkl} \dot{p}_{ki} V_l + m_i e_{jkl} P_{ki} \dot{v}_l + m_i e_{jkl} \omega_k e_{lmn} P_{mi} V_n + m_i e_{jkl} P_{ki} e_{lmn} \dot{\omega}_m P_{ni} + m_i e_{jkl} P_{ki} \ddot{p}_{li}$$

$$M_{g_j} = m_i e_{jkl} V_k \dot{p}_{li} + m_i e_{jkl} V_k e_{lmn} \omega_m P_{ni} - M_{G_j}$$

Elastic Deformation at ith Mass:

$$f_{ji} = m_i\left[\dot{v}_j + e_{jkl}\omega_k V_l + \ddot{p}_{ji} + e_{jkl}\dot{\omega}_k P_{li}\right] + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{ijkl} p_{lk} \quad (2.12c)$$

Equations (2.12) are expanded in equations (2.13) using the following definitions:

$$\vec{v} = u_1 \hat{i}_B + u_2 \hat{j}_B + u_3 \hat{k}_B = u \hat{i}_B + v \hat{j}_B + w \hat{k}_B \qquad \vec{V} = U_1 \hat{i}_B + U_2 \hat{j}_B + U_3 \hat{k}_B = U_0 \hat{i}_B + V_0 \hat{j}_B + W_0 \hat{k}_B$$

$$\vec{\omega} = \omega_1 \hat{i}_B + \omega_2 \hat{j}_B + \omega_3 \hat{k}_B = p \hat{i}_B + q \hat{j}_B + r \hat{k}_B \qquad \vec{\Omega} = \Omega_1 \hat{i}_B + \Omega_2 \hat{j}_B + \Omega_3 \hat{k}_B = P \hat{i}_B + Q \hat{j}_B + R \hat{k}_B$$

$$\vec{p}_i = p_{1i} \hat{i}_B + p_{2i} \hat{j}_B + p_{3i} \hat{k}_B \qquad \vec{P}_i = P_{1i} \hat{i}_B + P_{2i} \hat{j}_B + P_{3i} \hat{k}_B$$

$$(\vec{A} + \vec{a}) = \sum_{i=1}^{N} m_i \frac{\delta^2 \vec{P}_i}{\delta t^2} + \sum_{i=1}^{N} m_i \frac{\delta^2 \vec{p}_i}{\delta t^2} = (A_1 + a_1)\hat{i}_B + (A_2 + a_2)\hat{j}_B + (A_3 + a_3)\hat{k}_B$$

$$(\vec{B} + \vec{b}) = \sum_{i=1}^{N} m_i \frac{\delta \vec{P}_i}{\delta t} + \sum_{i=1}^{N} m_i \frac{\delta \vec{p}_i}{\delta t} = (B_1 + b_1)\hat{i}_B + (B_2 + b_2)\hat{j}_B + (B_3 + b_3)\hat{k}_B$$

$$(\vec{C} + \vec{c}) = \sum_{i=1}^{N} m_i \vec{P}_i + \sum_{i=1}^{N} m_i \vec{p}_i = (C_1 + c_1)\hat{i}_B + (C_2 + c_2)\hat{j}_B + (C_3 + c_3)\hat{k}_B$$

where u, v, w, p, q, r; Uo, Vo, Wo, P, O, R are the standard notation, such as defined in reference 10.

Rigid Body Translation:

$$
\begin{aligned}
f_1 - Mg(\Theta\cos\Theta_0) &= M(\dot{u} + qW_0 - rV_0) + \boxed{a_1} + \dot{q}C_3 - \dot{r}C_2 \\
f_2 + Mg\phi\cos\Theta_0 &= M(\dot{v} + rU_0 - pW_0) + \boxed{a_2} + \dot{r}C_1 - \dot{p}C_3 \\
f_3 - Mg(\Theta\sin\Theta_0) &= M(\dot{w} + pV_0 - qU_0) + \boxed{a_3} + \dot{p}C_2 - \dot{q}C_1
\end{aligned}
\qquad (2.13a)
$$

Rigid Body Rotation

$$
\begin{aligned}
M_{a1} &= \boxed{\begin{array}{l} W_0 b_2 + \dot{w}C_2 - V_0 b_3 - \dot{v}C_3 + M_{g1} \\ + q(V_0C_1 - U_0C_2) - r(U_0C_3 - W_0C_1) \end{array}} + I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r} + \boxed{\sum_{i=1}^{N} m_i P_{2i}\ddot{p}_{3i} - \sum_{i=1}^{N} m_i P_{3i}\ddot{p}_{2i}} \\
M_{a2} &= \boxed{\begin{array}{l} U_0 b_3 + \dot{u}C_3 - W_0 b_1 - \dot{w}C_1 + M_{g2} \\ + r(W_0C_2 - V_0C_3) - p(V_0C_1 - U_0C_2) \end{array}} - I_{yx}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r} + \boxed{\sum_{i=1}^{N} m_i P_{3i}\ddot{p}_{1i} - \sum_{i=1}^{N} m_i P_{1i}\ddot{p}_{3i}} \\
M_{a3} &= \boxed{\begin{array}{l} V_0 b_1 + \dot{v}C_1 - U_0 b_2 - \dot{u}C_2 + M_{g3} \\ + p(U_0C_3 - W_0C_1) - q(W_0C_2 - V_0C_3) \end{array}} - I_{zx}\dot{p} - I_{zy}\dot{q} + I_{zz}\dot{r} + \boxed{\sum_{i=1}^{N} m_i P_{1i}\ddot{p}_{2i} - \sum_{i=1}^{N} m_i P_{2i}\ddot{p}_{1i}}
\end{aligned}
\qquad (2.13b)
$$

Elastic Deformation at ith Mass:

$$
\begin{aligned}
f_{1i} - m_i g(\Theta\cos\Theta_0) &= m_i(\dot{u} + qW_0 - rV_0 + \dot{q}P_{3i} - \dot{r}P_{2i}) + m_i\ddot{p}_{1i} + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{i1kl}\,p_{lk} \\
f_{2i} + m_i g\phi\cos\Theta_0 &= m_i(\dot{v} + rU_0 - pW_0 + \dot{r}P_{1i} - \dot{p}P_{3i}) + m_i\ddot{p}_{2i} + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{i2kl}\,p_{lk} \\
f_{3i} - m_i g(\Theta\sin\Theta_0) &= m_i(\dot{w} + pV_0 - qU_0 + \dot{p}P_{2i} - \dot{q}P_{1i}) + m_i\ddot{p}_{3i} + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{i3kl}\,p_{lk}
\end{aligned}
\qquad (2.13c)
$$

where the product $K_{i1kl}\,p_{lk}$ is the stiffness force on the ith mass in the $\hat{i}_B$ direction (subscript 1) due to the vector displacement, p_{lk}, of the kth mass. Similarly for K_{i2kl} and K_{i3kl}.

The gravitational forces acting on the system appear explicitly in equations (2.13a) and (2.13c) and remain implicitly represented in equations (2.13c). Following the convention of most aeroelastic analysts, the terms "axis translation" and "axis rotation" are replaced by "rigid body translation" and "rigid body rotation". This less precise terminology has become standard even though the term "rigid" is certainly misleading in elastic aircraft analysis. Its use illustrates a commonly encountered problem - often one aeroelastic engineering discipline, e.g., flutter, loads, stability and control, etc., generates a unique terminology that appears inconsistent or non-precise to other engineering disciplines.

Up to this point in the development, the body fixed axis system has been "fixed" arbitrarily to the body, since the selection of an axis system often depends upon the problem of interest. However, in elastic aircraft analysis, an axis that facilitates the solution of equations (2.13) must be selected. Consider the axis attached to the center of mass of the flight vehicle. In this case the terms C_1, C_2, C_3 and all terms in the first box of equations (2.13b) are zero. If in addition, the axis that is attached to the center of mass is also the "mean axis" (15), those terms in the boxes in equations (2.13) are eliminated. The result is an immense simplification of equations (2.13): the only remaining coupling between the axis translation and rotation equations and the elastic deformation equations occurs in the aerodynamic terms f_j, f_{ji}, and M_{a_j}.

This paper will select the mean axis for subsequent analysis. Since the initial orientation of the mean axis is a user specified, this paper will select a mean axis orientation such that Θ_0 is zero and such that the products of inertia I_{xy} and I_{zy} are zero. Equations (2.13) become

$$
\begin{aligned}
f_1 - mg\theta &= M(\dot{u} + qW_0 - rV_0) \\
f_2 + mg\phi &= M(\dot{v} + rU_0 - pW_0) \\
f_3 &= M(\dot{w} + pV_0 - qU_0)
\end{aligned}
\qquad (2.14a)
$$

$$M_{a1} = I_{xx}\dot{p} - I_{xz}\dot{r}$$

$$M_{a2} = I_{yy}\dot{q}$$

$$M_{a3} = -I_{xz}\dot{p} + I_{zz}\dot{r} \tag{2.14b}$$

$$f_{1i} - m_i g\Theta = m_i(\dot{u} + qW_0 - rV_0 + \dot{q}P_{3i} - \dot{r}P_{2i}) + m_i\ddot{p}_{1i} + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{i1kl}\,p_{lk}$$

$$f_{2i} + m_i g\phi = m_i(\dot{v} + rU_0 - pW_0 + \dot{r}P_{1i} - \dot{p}P_{3i}) + m_i\ddot{p}_{2i} + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{i2kl}\,p_{lk}$$

$$f_{3i} = m_i(\dot{w} + pV_0 - qU_0 + \dot{p}P_{2i} - \dot{q}P_{1i}) + m_i\ddot{p}_{3i} + \sum_{k=1}^{N}\sum_{l=1}^{3} K_{i3kl}\,p_{lk} \tag{2.14c}$$

The kinematic relationships between the inertial axis and mean axis provides a relationship between the Euler angles ϕ , Θ , and ψ and the rotation rates p, q, and r:

$$\dot{\Theta} = q \ , \quad \dot{\phi} = p + r\,\cancelto{0.0}{\tan\Theta_0} \quad , \quad \dot{\psi} = r\,\cancelto{1.0}{\sec\Theta_0}$$

The combination of the kinematic relationships and the equations of the dynamics can be written in a compact matrix notation as in equations (2.15):

$$M(\dot{v} + M_{12}^{1}\dot{r}_{op} + \dot{M}_{12}^{2} r_{op}) = \bar{\Phi}_v^T f \tag{2.15a}$$

$$I_n\ddot{r}_{op} = \bar{\Phi}_r^T f \tag{2.15b}$$

$$K d_p + m_i(\ddot{d}_p + \bar{\Phi}(\dot{v}_p + M_1 v_p + M_2 r_{op}')) = f \tag{2.15c}$$

$$v^T = \lfloor u\ v\ w \rfloor \qquad r_{op}^T = \lfloor \phi\ \theta\ \psi \rfloor$$

$$d_p^T = \lfloor p_{11}\ p_{12}\ p_{13}\ \cdots\ p_{1N}\ p_{2N}\ p_{3N} \rfloor$$

$$v_p^T = \lfloor v^T\ \dot{r}_{op}^T \rfloor \qquad r_{op}'^{\,T} = \lfloor 0\ 0\ 0\ r_{op}^T \rfloor$$

$$K = [K_{ijkl}]$$

$$M_{12}^1 = \begin{bmatrix} 0 & W_0 & -V_0 \\ -W_0 & 0 & U_0 \\ V_0 & -U_0 & 0 \end{bmatrix} \qquad M_{12}^2 = \begin{bmatrix} 0 & g & 0 \\ -g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I_n = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix} \qquad M = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}$$

$$\bar{\Phi}_v = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{i=1} \\ \vdots \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{i=N} \end{bmatrix} \qquad \bar{\Phi}_r = \begin{bmatrix} \begin{bmatrix} 0 & P_{3i} & -P_{2i} \\ -P_{3i} & 0 & P_{1i} \\ P_{2i} & -P_{1i} & 0 \end{bmatrix}_{i=1} \\ \vdots \\ \begin{bmatrix} 0 & P_{3i} & -P_{2i} \\ -P_{3i} & 0 & P_{1i} \\ P_{2i} & -P_{1i} & 0 \end{bmatrix}_{i=N} \end{bmatrix}$$

$$\bar{\Phi} = [\bar{\Phi}_v\ \bar{\Phi}_r] \qquad m_i = \begin{bmatrix} m_1 & & & & & & 0 \\ & m_1 & & & & & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & & & & m_N & & \\ & & & & & m_N & \\ 0 & & & & & & m_N \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & M_{12}^1 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & M_{12}^2 \\ 0 & 0 \end{bmatrix}$$

$$f = A_1 v_p + A_2 \dot{v}_p + A_3 d_p + A_4 \dot{d}_p + A_5 \ddot{d}_p \tag{2.15d}$$

where A_1 and A_2 contain those aerodynamic force derivatives proportional to the rigid body translation and rotation.

A_3, A_4, A_5 contain those aerodynamic force derivatives proportional to the elastic deformations.

Combining equations (2.15a) and (2.15b) and summarizing equation (2.15c):

$$\underline{M}\left(\dot{v}_p + M_1 v_p + M_2 r'_{op}\right) = \bar{\Phi}^T f \tag{2.16a}$$

$$K d_p = -m_i\left(\ddot{d}_p + \bar{\Phi}\left(\dot{v}_p + M_1 v_p + M_2 r'_{op}\right)\right) + f \tag{2.16b}$$

where $\underline{M} = \begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix}$

Equations (2.16) and (2.15d) are the equations of motion for a controls fixed elastic aircraft. They describe the linear, small perturbation motion about a reference state (initial condition) of wings level, rectilinear flight. A comparison of equations (2.16) to equations (6.118) and (6.146) of reference 12 indicates an exact correspondence. Thus the Newtonian approach of reference 12 and the Lagrangian approach of this Memo have resulted in the same equations of motion. It should be noted that this mathematical representation of the aerodynamic forces due to rigid body and elastic motions, i.e., A_1 , A_2 , A_3 , A_4 , and A_5, is unique to reference 12 and this paper. The alternative method often found in the literature is a combination of A_1 with A_2 and of A_3 and A_4 with A_5 . This combination permits the use of the many unsteady aerodynamic theoretical methods currently found in the literature. The disadvantage of the combination is its implicit dependence upon the frequency (or time) variable. This implicit dependence requires a costly iterative frequency solution for the aerodynamic forces due to the rigid body motions v_p and the elastic motions d_p prior to or simultaneously with the determination of the eigenvalues of the equations of motion of the elastic aircraft. An iterative frequency solution is not required for the aerodynamic forces due to the v_p and d_p described in this paper. However, this computational savings currently limits the validity of the solutions of this paper and reference 12 to the "low frequency" eigenvalues. The upper frequency boundary (Figure 2.3) of the validity of the "low frequency" approximation has not been determined, but from the practical viewpoint, it is probably sufficient to do most current integrated control system designs.

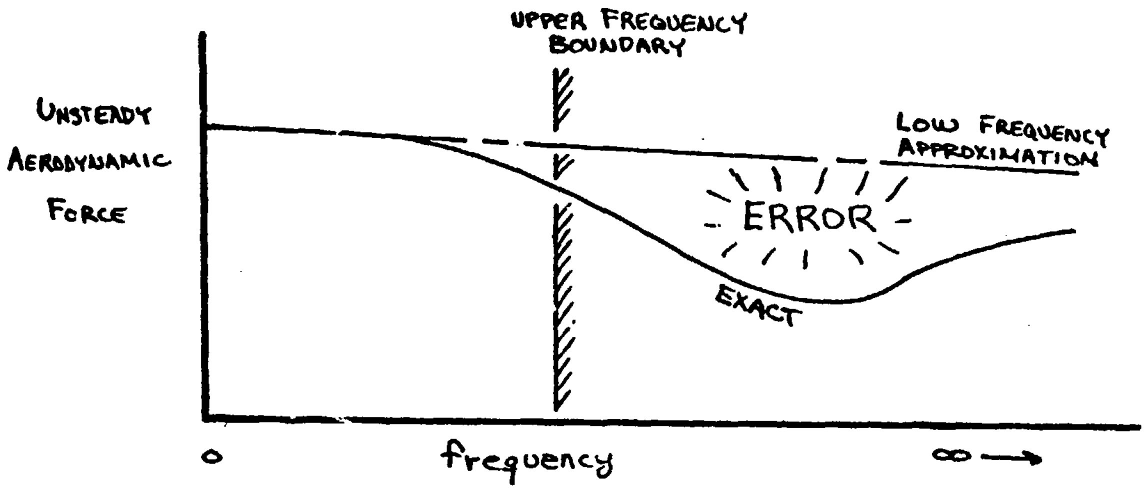


Figure 2.3 Comparison of Low Frequency Approximation to Exact, Unsteady Aerodynamic Theory

2.3 Controls-Free, Elastic Aircraft Equations of Motion - EXACT Formulation

A controls-free, elastic aircraft analysis problem is characterized by active aerodynamic control surface motion in response to pilot or augmentation system commands. Two common reasons for these necessary commands are atmospheric gusts and atmospheric turbulence. The equations for the controls-free analysis are not as determinable as equations (2.16) and (2.15d) for the controls-fixed analysis. The reason for this is the dependence of the turbulence and gust aircraft responses to the geometry and the flight Mach number of the aircraft. This difficulty can be overcome by assuming a general form for the additional terms to be added to equations (2.15d). The net results, equations (2.18), or (2.19), are termed the EXACT formulation.

2.3.1 Structural damping

In the case to be developed in this section. it will be assumed that structural damping is viscous in nature and proportional to elastic displacement rate, $\dot{d}_p$. The damping force is represented as $D\dot{d}_p$, where D is a damping matrix and, in general, is a dense array. It should be noted that the "usual assumption" that D is proportional to m_i or k "has not been made, thus, the effect of these "usual assumption" can then be identified in the development of the special formulations in Section 3.0. In the event that D is proportional to m_i or K , the following identies are valid:

$$\bar{\phi}^T D \phi \equiv 0 \quad (2.17a)$$

$$\phi^T D \phi \equiv [d] \quad (2.17b)$$

$$\phi^T D \bar{\phi} \equiv 0 \quad (2.17c)$$

2.3.2 Atmospheric gusts and turbulence

The representation of the atmospheric gust or the atmospheric turbulence often depends upon the configuration being analyzed. Commercial and military aircraft are designed using different gust and turbulence mathematical models. A given military aircraft may be designed to satisfy several gust and turbulence mathematical models, depending upon the extent of its flight profile. It will be assumed that both the gust and turbulence can be represented as 3 components of force on each of the N masses. Also, it will be assumed that gust forces f_g and turbulence forces f_t have the same element ordering as aerodynamic forces f . The total moments and forces applied to the rigid body are then:

$$\bar{\phi}^T f_g = \begin{Bmatrix} f_{1g} \\ f_{2g} \\ f_{3g} \\ M_{1g} \\ M_{2g} \\ M_{3g} \end{Bmatrix} \qquad \bar{\phi}^T f_t = \begin{Bmatrix} f_{1t} \\ f_{2t} \\ f_{3t} \\ M_{1t} \\ M_{2t} \\ M_{3t} \end{Bmatrix} \quad (2.17d)$$

The general form of f_g and f_t allows several different gust forms, i.e., ramps, steps, 1-cos, sin, etc. and several different turbulence forms, i.e., 1, 2, or 3 component Von Karman or Dryden power spectral density representations. It is only required that the spatially defined gust and turbulence be transformed to the body fixed mean axis prior to inclusion into equations (2.15d) as either f_g or f_t . Typical transformation considerations may be found in reference (25).

2.3.3 Active aerodynamic control surfaces

The forces due to specified number of active aerodynamic control surfaces, c in number, can also be defined in a general sense. The three components of control surface induced forces on each of the N masses is defined to be f_c. Again, the elements of f_c are arranged in the same order as those in f. It will be assumed that the elements in f_c are proportional to δ_c, $\dot{\delta}_c$, and $\ddot{\delta}_c$ by an equation to be developed for each specialized application, i.e.,

$$\{f_c\} = \{A_{6i}\delta_i + A_{7i}\dot{\delta}_i + A_{8i}\ddot{\delta}_i\}_{i=1} + \cdots + \{A_{6i}\delta_i + A_{7i}\dot{\delta}_i + A_{8i}\ddot{\delta}_i\}_{i=c} \quad (2.17e)$$

Inserting the general representation of structural damping, atmospheric gust and turbulence, and active aerodynamic control surfaces, results in equations (2.18):

$$\underline{M}(\dot{v}_p + M_1 v_p + M_2 r_{op}') = \bar{\Phi}^T (f + f_c + f_g + f_t) \quad (2.18a)$$

$$D\dot{d}_p + K d_p = -m_i(\ddot{d}_p + \bar{\Phi}(\dot{v}_p + M_1 v_p + M_2 r_{op}'))$$

$$+ f + f_c + f_g + f_t \quad (2.18b)$$

where $f = A_1 v_p + A_2 \dot{v}_p + A_3 d_p + A_4 \dot{d}_p + A_5 \ddot{d}_p$ (2.18c)

Alternatively:

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ m_i\bar{\Phi}_v & m_i\bar{\Phi}_r & m_i \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \\ \ddot{d}_p \end{Bmatrix} + \begin{bmatrix} MM_{12}^1 & 0 \\ 0 & 0 \\ m_i\bar{\Phi}_v M_{12}^1 & D \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{d}_p \end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0 \\ 0 & 0 \\ m_i\bar{\Phi}_v M_{12}^2 & K \end{bmatrix} \begin{Bmatrix} r_{op} \\ d_p \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \\ f_i \end{Bmatrix}$$

(2.19a)

where

$$f_i = f + f_c + f_g + f_t$$

$$= A_1 v_p + A_2 \dot{v}_p + A_3 d_p + A_4 \dot{d}_p + A_5 \ddot{d}_p + f_c + f_g + f_t \quad (2.19b)$$

Equations (2.19) are defined to be an EXACT formulation of the equations of motion of a controls-free, elastic aircraft. Inspection of these equations indicates their form to be linear ordinary differential equations requiring a simultaneous solutions for the 3N+6 unknowns υ_p and d_p subject to the 6 constraints imposed by mean axis attached to the center of mass of the aircraft.

It is particularly important to note that the solution of equations (2.19) will result in both complex number eigenvalues and eigenvectors:

- The real and imaginary portions of the eigenvalues are a measure of the stability, circular frequency, and phase difference of the eigenvectors.

- The real and imaginary portions of the eigenvectors reflect the spatial orientation of the masses associated with an eigenvalue, i.e., the masses of the ith mode shapes do not have precisely 0° or 180° phase difference in position when the ith eigenvector is complex (reference 26 and 27).

It will be shown in the next section that this result from equations (2.19) is different than that currently calculated by the "flutter" and "dynamic load" engineering disciplines. If the flight control engineer chooses the typical "flutter" equations of motion in which the real number mode shapes are used, the phase and gain relationships of any control system that is synthesized may be in error. The magnitude of this error depends upon the relative magnitude of the aerodynamic terms and structural damping terms due to the elastic motion.

3.0 APPROXIMATE FORMULATIONS

The solution of the EXACT formulation requires a large digital computer. A key element in the computerized solution is the routine that determines the complex number eigenvalues and eigenvectors. Presently, these routines can calculate a "small number" of complex number eigenvalues and eigenvectors with a high degree of accuracy. As the number of complex number eigenvalues increases, the accuracy of the routines deteriorates due to the limited number of significant digits that the computer can retain. In the case of a solution to the EXACT formulation, even those calculations using double precision will eventually suffer, since a typical value of an N is 300 or approximately 900 eigenvalues. Consequently, an accurate solution of the EXACT formulation of the equations of motion is unrealizable for all but simplified analysis problems.

In this section, five approximations to the EXACT formulation of the equations of motion for the controls-free aircraft are constructed to permit a cheaper and faster determination of the unknown motions. These approximations are developed by reducing the number of unknowns and equations through the use of assumptions that may be valid only for a specific elastic airplane problem. In a mathematical sense, there is no single set of these approximate equations that are valid for all aircraft or a single aircraft at all flight conditions. In an engineering sense, one or more approximate sets of equations may be valid. In each of the five formulations, a matrix equation similar to the EXACT formulation, equation (2.19), will be developed as the final result.

The reduction in the number of the equations and unknowns often results in equations of motion that are from 10 to 50 in number. This order of reduction, e.g., from 100's to 10's in number is sufficient to overcome the numerical difficulty in the digital computer routines used to calculate the complex number eigenvalues and eigenvectors. If the reduction in equation number and complexity is large and excess central memory storage space becomes available, the nonlinear aerodynamic and structural data may be incorporated into the analysis. These nonlinear terms are not considered explicitly in equations (2.19), but may be easily included[12].

There are two means to effect the simplification of the EXACT formulation:

1. The eigenvalue solution of the EXACT formulation equation can be truncated at some selected frequency. The effects of the neglected higher frequency eigenvalues must then be "filtered" from the integrated control system.

2. The EXACT formulation equations can be simplified to eliminate the complexity by reducing the number of equations and unknowns, <u>prior</u> to

their solution.

There is no engineering experience at the present time that would indicate which of the two simplifications is the more appropriate for elastic vehicles at all M-q flight conditions.

The second simplification is that most commonly employed and reported in the literature. When this simplification is selected, the flight control engineer must decide upon a static or dynamic aeroelastic formulation appropriate to his elastic aircraft. This decision requires a certain amount of engineering insight based upon experimentation and previous experience. It is the second simplification that will be discussed for the remainder of this paper; the first simplification is currently being investigated by the AFFDL/FGC.

The reduced set of equations to be considered in this Section are:

- QUASI STATIC
- MODAL SUBSTITUTION
- RESIDUAL STIFFNESS
- RESIDUAL FLEXIBILITY
- MODAL TRUNCATION

3.1 QUASI STATIC Formulation

The QUASI STATIC formulation is the classical method used by stability and control and flight control engineers prior to the advent of highly flexible aircraft. This formulation is the most rapid and the cheapest to solve. Due to the small number of equations, it is ideally suited to a nonlinear aerodynamics analysis based upon experimentally measured data. An illustration of this approach is found in reference 12.

The derivation of the QUASI STATIC equations begin with equations (2.19):

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ m_i\bar{\Phi}_v & m_i\bar{\Phi}_r & m_i \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \\ \ddot{d}_p \end{Bmatrix} + \begin{bmatrix} MM_{12}' & 0 \\ 0 & 0 \\ m_i\bar{\Phi}_v M_{12}' & D \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{d}_p \end{Bmatrix} + \begin{bmatrix} MM_{12}' & 0 \\ 0 & 0 \\ m_i\bar{\Phi}_v M_{12}' & K \end{bmatrix} \begin{Bmatrix} r_{op} \\ d_p \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \\ f_i \end{Bmatrix} \tag{2.19a}$$

where

$$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 d_p + A_4 \dot{d}_p + A_5 \ddot{d}_p + f_c + f_g + f_t \tag{2.19b}$$

These equations are simplified with assumptions A1 and A2

A1 the aerodynamic forces proportional to $\dot{d}_p$ and $\ddot{d}_p$ are zero i.e., $A_4 \dot{d}_p = A_5 \ddot{d}_p = 0$.

A2 the structural inertial and damping forces are zero, i.e., $m \ddot{d}_p = D \dot{d}_p = 0$.

Equation (2.19) becomes:

$$\begin{bmatrix} M & 0 \\ 0 & I_n \\ m_i \bar{\Phi}_v & m_i \bar{\Phi}_r \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \end{Bmatrix} + \begin{bmatrix} MM_{12}^1 \\ 0 \\ m_i \bar{\Phi}_v M_{12}^1 \end{bmatrix} \left\{ \dot{r}_{op} \right\} + \begin{bmatrix} MM_{12}^2 & 0 \\ 0 & 0 \\ m_i \bar{\Phi}_v M_{12}^2 & K \end{bmatrix} \begin{Bmatrix} r_{op} \\ d_p \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \\ f_i \end{Bmatrix}$$

Alternatively:

$$\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \end{Bmatrix} + \begin{bmatrix} MM_{12}^1 \\ 0 \end{bmatrix} \left\{ \dot{r}_{op} \right\} + \begin{bmatrix} MM_{12}^2 \\ 0 \end{bmatrix} \left\{ r_{op} \right\} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \end{Bmatrix} \tag{3.1a}$$

$$m_i \bar{\Phi}_v \dot{v} + m_i \bar{\Phi}_r \ddot{r}_{op} + m_i \bar{\Phi}_v M_{12}^1 \dot{r}_{op} + m_i \bar{\Phi}_v M_{12}^2 r_{op} + K d_p = f_i \tag{3.1b}$$

where now

$$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 d_p + f_c + f_g + f_t \tag{3.1c}$$

Substitute f_i from equation (3.1c) into equation (3.1b), collect the common terms, and then determine d_p :

$$m_i \bar{\Phi} \dot{v}_p + m_i \bar{\Phi} M_1 v_p + m_i \bar{\Phi} M_2 r'_{op} + K d_p = f_i$$

$$= A_1 v_p + A_2 \dot{v}_p + A_3 d_p + f_c + f_g + f_*$$

$$d_p = [K - A_3]^{-1} \{ (A_2 - m_i \bar{\Phi}) \dot{v}_p + (A_1 - m_i \bar{\Phi} M_1) v_p - (m_i \bar{\Phi} M_2) r'_{op} + f_c + f_g + f_* \} \quad (3.2)$$

Substitute d_p of equation (3.2) into equation (3.1c) and group common terms:

$$f_i = (A_1 + A_3 (K - A_3)^{-1} (A_1 - m_i \bar{\Phi} M_1)) v_p + (A_2 + A_3 (K - A_3)^{-1} (A_2 - m_i \bar{\Phi})) \dot{v}_p - A_3 (K - A_3)^{-1} (m_i \bar{\Phi} M_2) r'_{op} + (I + A_3 (K - A_3)^{-1})(f_c + f_g + f_*) \quad (3.3)$$

The QUASI STATIC formulation is presented in equations (3.4):

$$\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \end{Bmatrix} + \begin{bmatrix} M M_{12}^{1} \\ 0 \end{bmatrix} \{ \dot{r}_{op} \} + \begin{bmatrix} M M_{12}^{2} \\ 0 \end{bmatrix} \{ r_{op} \} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \end{Bmatrix} \quad (3.4a)$$

$$f_i = (I + A_3 (K - A_3)^{-1}) (A_1 v_p + A_2 \dot{v}_p + f_c + f_g + f_*) + A_3 (K - A_3)^{-1} (-m_i \bar{\Phi} \dot{v}_p - m_i \bar{\Phi} M_1 v_p - m_i \bar{\Phi} M_2 r'_{op}) \quad (3.4b)$$

All explicit reference to elastic motion d_p has been eliminated in equation (3.4). The factor $(I+A_3(K-A_3)^{-1})$ is a "quasi static" aero-elastic correction factor applied to the rigid airplane forces and moments. The factors $-m_i\bar{\Phi}$, $-m_i\bar{\Phi}M_1$, and $-m_i\bar{\Phi}M_2$ are "inertial relief" corrections.

An alternate formulation exists in the literature. It is based upon the availability of a flexibility matrix for the cantilevered structure and the construction of the "free-free" flexibility matrix $\bar{C}$. This formulation is repeated below from reference 12. Since the formulation in reference 12 describes motion of a controls-fixed elastic aircraft in quiescent air, editorial prerogative has been assumed and the terms $f_c + f_g + f_*$ have been added to the equations.

$$\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix}\begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \end{Bmatrix} + \begin{bmatrix} MM_{12}^1 \\ 0 \end{bmatrix}\begin{Bmatrix} \dot{r}_{op} \end{Bmatrix} + \begin{bmatrix} MM_{12}^2 \\ 0 \end{bmatrix}\begin{Bmatrix} r_{op} \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \end{Bmatrix}$$

$$f_i = [I - A_3\bar{C}]^{-1}\Big(A_1 v_p + A_2\dot{v}_p + f_c + f_g + f_* - A_3\bar{C}m_i\bar{\Phi}(\dot{v}_p + M_1 v_p + M_2 r'_{op})\Big)$$

where $\bar{C} = [I - \bar{\Phi}\underline{M}^{-1}\bar{\Phi}^T m_i]C$, $\underline{M} = \begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix}$

C is the flexibility matrix for the cantilevered structure.

An individual applying the QUASI STATIC formulation has in effect assumed that the elastic deformation occurs instantaneously and in phase with the axis system motions. This representation is usually valid for the design of handling quality and reduced static stability augmentation systems of elastic aircraft at very low frequencies, or for elastic air-craft having a large frequency separation of its rigid body and elastic deformation motions. The formulation is not generally valid for the design of the other CCV-type control systems.

The QUASI STATIC formulation has as its advantages:

1. The solution of only 6 equations, permitting many M-q flight conditions to be rapidly analyzed.

2. The inclusion of nonlinear aerodynamic effects to flight controls

analyses using aeroelastic correction factors, e.g., $C_{L_\alpha}^{ELAS}/C_{L_\alpha}^{RIGID}$, $C_{m_{\delta_e}}^{ELAS}/C_{m_\delta}^{RIGID}$, etc.

3. The convenient implementation on limited size analog or digital flight simulators.

3.2 MODAL SUBSTITUTION

The MODAL SUBSTITUTION formulation is classified as "dynamic aeroelastic", since the elastic motion is not assumed to be in phase with the rigid body motions. It is assumed that:

A3 structural damping forces are negligible, i.e., very small compared to the structural stiffness and structural inertial forces.

A4 aerodynamic forces due to the elastic deformation are negligible, i.e., small compared to the structural stiffness and structural inertial forces.

Assumptions A3 and A4 permit a coordinate transformation using the invacuum, orthogonal modes of vibration. This substitution results in the simplification of the complex number eigenvector solutions of equations (2.19) to real number eigenvectors. The procedure for accomplishing this transformation is:

1. Recall equations (2.18b):

$$D\dot{d}_p + Kd_p = -m_i\left(\ddot{d}_p + \bar{\Phi}(\dot{v}_p + M_1 v_p + M_2 r'_{op})\right) + f + f_c + f_g + f_* \qquad (2.18b)$$

2. Represent d_p as an ensemble of invacuum, orthogonal modes of vibration, i.e., $d_p = \phi u$, where $\bar{\Phi}^T m_i \phi = \phi^T m_i \bar{\Phi} = \bar{\Phi}^T K \phi = 0$.

3. Post multiply equations (2.18b) by ϕ^T.

4. Apply assumptions A3 and A4.

The result is presented in equations (3.5):

$$\phi^T m_i \phi \ddot{u} + \phi^T K \phi u = -\phi^T m_i \bar{\Phi}(\cancel{\dot{v}_p + M_1} v_p + M_2 r'_{op}) \xrightarrow{} \text{ZERO, ORTHOGONALITY}$$
$$+ \phi^T (f \cancel{+ f_c +} f_g + f_*) \xrightarrow{} \text{ASSUMPTION A4}$$
$$- \phi^T \cancel{D} \phi \dot{u} \xrightarrow{} \text{ASSUMPTION A3}$$

$$m\ddot{u} + ku = 0 \qquad (3.5)$$

where $$\phi^T m_i \phi = m \tag{3.6a}$$

$$\phi^T K \phi = k \tag{3.6b}$$

5. Determine the eigenvalues and eigenvectors satisfying equation (3.5) such that (3.6a) and (3.6b) are valid, i.e., find the eigenvalues of the determinate $|[I\lambda^2 + m^{-1}k]| = 0$. This eigenvalue problem is well documented in reference 28.

Of the λ_i, in the general 6D case, 3N-6 in number are real non-zero numbers; those λ_i that are real are used to determine the value of ϕ. Six of the λ_i are zero due to the use of the stiffness matrix k for the "free-free" structure. Physically, the zero eigenvalues mean that the elastic displacements are known within an arbitrary 3 rotations and 3 translations of the elastic body ensemble. The amplitudes of the translations and rotations are zero by definition of the mean axis. The degrees of freedom may thus be removed by combining equation (3.5) with equations (2.18b) and (2.18c):

$$\underline{M}\,(\dot{v}_p + M_1 v_p + M_2 r'_{op}) = \bar{\Phi}^T (f + f_c + f_g + f_t) \tag{3.7a}$$

$$m\ddot{u} + ku = \phi^T(f + f_c + f_g + f_t) - \phi^T D \phi \dot{u} \tag{3.7b}$$

$$f = A_1 v_p + A_2 \dot{v}_p + A_3 \phi u + A_4 \phi \dot{u} + A_5 \phi \ddot{u} \tag{3.7c}$$

Note that this coordinate transformation of d_p to ϕu has reduced the number of equations and unknowns.

Equations (3.7) can be rewritten into the matrix format similar to equations (2.19):

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \\ \ddot{u} \end{Bmatrix} + \begin{bmatrix} MM_{12}^1 & 0 \\ 0 & 0 \\ 0 & \phi^T D \phi \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{u} \end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0 \\ 0 & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} r_{op} \\ u \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \\ \phi^T f_i \end{Bmatrix} \tag{3.8a}$$

where $$f_i = f + f_c + f_g + f_t \tag{3.8b}$$

The equations (3.8) are termed the MODAL SUBSTITUTION formulation. This formulation has reduced the complexity of the problem, i.e., complex number eigenvectors in equation (2.19) are reduced to prescribed real number eigenvectors in (3.8). However, this reduction has not eliminated the problem of determining a large number of complex eigenvalues and complex number eigenvector magnitudes. The "inaccurate" complex number eigenvalue routines required for the EXACT formulation, must still be used.

3.3 RESIDUAL STIFFNESS Formulation

The RESIDUAL STIFFNESS formulation reduces the number of equations and unknowns that are associated with the elastic motion in the MODAL SUBSTITUTION formulation. This reduction may be limited to only one mode shape or to most of the 3N-6 mode shapes. The reduction is accomplished by the engineer:

1. Noting that a large separation in the frequencies of the mode shapes has occurred.

2. Deciding that the problem need include only those eigenvalues that are less than some selected frequency.

The RESIDUAL STIFFNESS formulation then represents dynamically all the modes retained; it "statically" represents all the modes deleted.

First, separate d_p into retained and deleted modes:

$$d_p = [\phi_1 \; \phi_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \phi_1 u_1 + \phi_2 u_2 \tag{3.9}$$

where $\phi_1 u_1$ are modes to be retained.

$\phi_2 u_2$ are modes to be deleted.

The d_p of equations (3.9) is substituted into equation (3.8):

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \dot{r}_{op} \\ \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} MM_{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \phi_1^T D \phi_1 & \phi_1^T D \phi_2 \\ 0 & \phi_2^T D \phi_1 & \phi_2^T D \phi_2 \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{Bmatrix} r_{op} \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \\ \phi_1^T f_i \\ \phi_2^T f_i \end{Bmatrix} \tag{3.10a}$$

where $m_1 = \phi_1^T m_i \phi_1 \qquad k_1 = \phi_1^T K \phi_1$

$m_2 = \phi_2^T m_i \phi_2 \qquad k_2 = \phi_2^T K \phi_2$

$$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 (\phi_1 u_1 + \phi_2 u_2) + A_4 (\phi_1 \dot{u}_1 + \phi_2 \dot{u}_2) + A_5 (\phi_1 \ddot{u}_1 + \phi_2 \ddot{u}_2) + f_c + f_g + f_r \tag{3.10b}$$

In order to reduce equations (3.10) to the RESIDUAL STIFFNESS formulation, the following assumptions are required:

A5 the aerodynamic forces due to deleted modal velocity and acceleration are zero, i.e., $A_4\phi_2\dot{u}_2 = A_5\phi_2\ddot{u}_2 = 0$.

A6 the structural damping and inertial forces due to deleted modal deformation u_2 are zero, i.e., $\phi_1^TD\phi_2 = \phi_2^TD\phi_2 = 0$, $m_2\ddot{u}_2 = 0$.

A7 the structural damping of the retained modes on the deleted modes is zero, i.e., $\phi_2^TD\phi_1 = 0$.

Applying the assumptions to equations (3.10) reduces them to the form of equations (3.11):

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & m_1 \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \\ \ddot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM_{12}^1 & 0 \\ 0 & 0 \\ 0 & \phi_1^TD\phi_1 \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0 \\ 0 & 0 \\ 0 & k_1 \end{bmatrix} \begin{Bmatrix} r_{op} \\ u_1 \end{Bmatrix} = \begin{Bmatrix} \bar{\phi}_v^T f_i \\ \bar{\phi}_r^T f_i \\ \phi_1^T f_i \end{Bmatrix} \tag{3.11a}$$

$$k_2 u_2 = \phi_2^T f_i \tag{3.11b}$$

$$f_i = A_1 v_p + A_2\dot{v}_p + A_3\phi_1 u_1 + A_3\phi_2 u_2 + A_4\phi_1\dot{u}_1 + A_5\phi_1\ddot{u}_1 + f_c + f_g + f_t \tag{3.11c}$$

Equation (3.11b) is solved for u_2 and the result is substituted into equations (3.11c):

$$u_2 = k_2^{-1}\phi_2^T f_i$$
$$f_i = A_1 v_p + A_2\dot{v}_p + A_3\phi_1 u_1 + A_4\phi_1\dot{u}_1 + A_5\phi_1\ddot{u}_1 + A_3\phi_2 k_2^{-1}\phi_2^T f_i + f_c + f_g + f_t \tag{3.12a}$$

Equation (3.12a) has a common factor, f_i, appearing on both the right and the left side of the equality. A grouping of this factor to the left side of the equality and removing the multiplication factor by inversion reduces equation (3.12a) to the equation (3.12b)

$$f_i = [I - A_3\phi_2 k_2^{-1}\phi_2^T]^{-1}(A_1 v_p + A_2\dot{v}_p + A_3\phi_1 u_1 + A_4\phi_1\dot{u}_1 + A_5\phi_1\ddot{u}_1 + f_c + f_g + f_t) \tag{3.12b}$$

Summarizing equation (3.12b) and equation (3.11a):

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & m_1 \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \\ \ddot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM_{12}^1 & 0 \\ 0 & 0 \\ 0 & \phi_1^T D \phi_1 \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0 \\ 0 & 0 \\ 0 & k_1 \end{bmatrix} \begin{Bmatrix} r_{op} \\ u_1 \end{Bmatrix} = \begin{Bmatrix} \Phi_v^T f_i \\ \Phi_r^T f_i \\ \phi_1^T f_i \end{Bmatrix} \tag{3.13a}$$

$$f_i = [I - A_3 \phi_2 k_2^{-1} \phi_2^T]^{-1} \left(A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 + f_c + f_g + f_t \right) \tag{3.13b}$$

Equations (3.13) are the RESIDUAL STIFFNESS formulation. They are 6 + r in number, where r is the number of retained modes, ϕ_1 . A substantial simplification of the MODAL SUBSTITUTION formulation (equations (3.8)) has been accomplished. The 3N number of equations and unknowns has been reduced to 6+r number of equations and unknowns. The cost of the simplification is that the elastic airplane must satisfy assumptions A3 and A4 as in the MODAL SUBSTITUTION, and in addition, must satisfy assumptions A5, A6 , and A7 . The advantage of this technique is that r can be selected such that the 6+r equations are less than 50, thus, the "inaccuracy" of the complex number eigenvalue computer routine can be minimized. The disadvantage of the technique is that all the mode shapes, ϕ_1 and ϕ_2, must be calculated.

3.4 RESIDUAL FLEXIBILITY Formulation

The RESIDUAL FLEXIBILITY formulation eliminates the necessity to calculate ϕ_2 required in equation (3.13) of the RESIDUAL STIFFNESS formulation. The elimination is accomplished by redeveloping equations (2.18a) using the "free-free" flexibility matrix. An excellent description of this formulation is found in references 12 and 16. The expression equivalent to equation (2.18a) is equation (6.129) of reference 12:

$$d_p = -\bar{C} m_i \left(\ddot{d}_p + \Phi (\dot{v}_p + M_1 v_p + M_2 \dot{r}_{op}') \right) - \bar{C} D \dot{d}_p + \bar{C} f_i \tag{6.129, Ref. 12}$$

where $\bar{C} = [I - \Phi \underline{M}^{-1} \Phi^T m_i] C$ is the "free-free" flexibility matrix and C is the flexibility matrix for the aircraft structure restrained (at the c.g.) against rigid body translation and rotation. The terms $\bar{C} D \dot{d}_p + \bar{C} f_i$ have been added using editorial prerogative.

The removal of $\phi_2 k_2^{-1} \phi_2^T$ from equation (3.12a) is accomplished in

5 computational steps:

1. Recall equation (2.18a); apply A3 and A4 and substitute $\delta_p = \phi_1 u_1 + \phi_2 u_2$ from equation (3.9):

$$k u = \phi^T(-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) \cancel{- \phi^T m_i \Phi (\dot{v}_p + M_1 v_p + M_2 r'_{op})} \quad \text{ORTHOGONALITY}$$

$$k u = \phi^T(-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i)$$

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \phi_1^T \\ \phi_2^T \end{bmatrix} (-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} k_1^{-1} & 0 \\ 0 & k_2^{-1} \end{bmatrix} \begin{bmatrix} \phi_1^T \\ \phi_2^T \end{bmatrix} (-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) \tag{3.14}$$

2. Premultiply equation (3.14) by $[\phi_1\ \phi_2]$

$$[\phi_1\ \phi_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \delta_p = [\phi_1\ \phi_2] \begin{bmatrix} k_1^{-1} & 0 \\ 0 & k_2^{-1} \end{bmatrix} \begin{bmatrix} \phi_1^T \\ \phi_2^T \end{bmatrix} (-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i)$$

$$\delta_p = [\phi_1 k_1^{-1} \phi_1^T + \phi_2 k_2^{-1} \phi_2^T](-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) \tag{3.15}$$

3. Equate equation (2.36), reference 15 to equation (3.15), i.e.,

$$\delta_p \big|_{\text{eq(3.15)}} = \delta_p \big|_{\text{eq(6.129, ref. 12)}}$$

$$[\phi_1 k_1^{-1} \phi_1^T + \phi_2 k_2^{-1} \phi_2^T](-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) = -\bar{\mathcal{C}} D\dot{\delta}_p + \bar{\mathcal{C}} f_i - \bar{\mathcal{C}} m_i \ddot{\delta}_p - \bar{\mathcal{C}} m_i \Phi(\dot{v}_p + M_1 v_p + M_2 r'_{op})$$

Collecting terms that contain $\phi_2 k_2^{-1} \phi_2^T$:

$$\begin{aligned}\phi_2 k_2^{-1} \phi_2^T (-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) &= [\bar{C} - \phi_1 k_1^{-1} \phi_1^T](-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) \\ &\quad - \bar{C} m_i \bar{\Phi} (\dot{v}_p + M_1 v_p + M_2 r_{op}')\end{aligned}$$

Defining $R = \bar{C} - \phi_1 k_1^{-1} \phi_1^T$:

$$\begin{aligned}\phi_2 k_2^{-1} \phi_2^T (-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) &= R(-m_i \ddot{\delta}_p - D\dot{\delta}_p + f_i) \\ &\quad - \bar{C} m_i \bar{\Phi} (\dot{v}_p + M_1 v_p + M_2 r_{op}')\end{aligned} \tag{3.16}$$

4. Substitute $\delta_p = \phi_1 u_1 + \phi_2 u_2$ from equations (3.9) into equation (3.16) and recall A5 , A6 , A7 :

$$\begin{aligned}[\phi_2 k_2^{-1} \phi_2^T] \left\{ - \begin{bmatrix} m_i \phi_1 & 0 \\ 0 & m_i \phi_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} - \begin{bmatrix} D\phi_1 & 0 \\ 0 & D\phi_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \{f_i\} \right\} \\ = [R] \left\{ - \begin{bmatrix} m_i \phi_1 & 0 \\ 0 & m_i \phi_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} - \begin{bmatrix} D\phi_1 & 0 \\ 0 & D\phi_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \{f_i\} \right\} \\ - [\bar{C}][m_i][\bar{\Phi}] \left\{ \dot{v}_p + M_1 v_p + M_2 r_{op}' \right\}\end{aligned}$$

$$\begin{aligned}[\phi_2 k_2^{-1} \phi_2^T](\overset{\text{ORTHOGONALITY}}{-\cancel{m_i \phi_1 \ddot{u}_1}} \overset{\text{ASSUMPTION A7}}{- \cancel{D\phi_1 \dot{u}_1}} + f_i) \\ = R(-m_i \phi_1 \ddot{u}_1 - D\phi_1 \dot{u}_1 + f_i) \\ - \bar{C} m_i \bar{\Phi} (\dot{v}_p + M_1 v_p + M_2 r_{op}')\end{aligned} \tag{3.17}$$

Equation (3.17) becomes:

$$[\phi_2 k_2^{-1} \phi_2^T] f_i = R\,(-m_i \phi_1 \ddot{u}_1 - D\phi_1 \dot{u}_1 + f_i) \\ - \bar{C} m_i \bar{\Phi}\,(\dot{v}_p + M_1 v_p + M_2 r_{op}') \qquad (3.18)$$

5. Recall equation (3.12a):

$$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 + f_g + f_c + f_t \\ + A_3 \phi_2 k_2^{-1} \phi_2^T f_i \qquad (3.12a)$$

Substitute equation (3.18) into equation (3.12a), collect the terms on left side of equation, and remove the common term by inversion:

$$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 + f_g + f_t + f_c \\ + A_3 R(-m_i \phi_1 \ddot{u}_1 - D\phi_1 \dot{u}_1 + f_i) \\ - A_3 \bar{C} m_i \bar{\Phi}\,(\dot{v}_p + M_1 v_p + M_2 r_{op}')$$

$$f_i = [I - A_3 R]^{-1}\,\big(A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 \\ + f_c + f_g + f_t + A_3 R\,(-m_i \phi_1 \ddot{u}_1 - D\phi_1 \dot{u}_1) \\ - A_3 \bar{C} m_i \bar{\Phi}\,(\dot{v}_p + M_1 v_p + M_2 r_{op}')\big) \qquad (3.19a)$$

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & m_1 \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \\ \ddot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM_{12}' & 0 \\ 0 & 0 \\ 0 & \phi_1^T D \phi_1 \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0 \\ 0 & 0 \\ 0 & k_1 \end{bmatrix} \begin{Bmatrix} r_{op} \\ u_1 \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v^T f_i \\ \bar{\Phi}_r^T f_i \\ \phi_1^T f_i \end{Bmatrix}$$

(3.19b)

Equations (3.19) are the RESIDUAL FLEXIBILITY equations that describe the motion of a controls-free elastic airplane. As in the RESIDUAL STIFFNESS formulation, the number of unknowns is 6+r, where r is the number of retained modes ϕ_1 . This formulation does not depend upon $\phi_2 k_2^{-1} \phi_2^T$ as did the RESIDUAL STIFFNESS formulation, thus, ϕ_2 need not be calculated. The penalty for using the RESIDUAL FLEXIBILITY formulations are the necessities to:

1. Determine C for the aircraft.

2. Calculate more numerous matrix products in f_i .

The relative cost advantage of the RESIDUAL STIFFNESS and RESIDUAL FLEXIBILITY formulations is unknown at this time and is being investigated further.

3.5 MODAL TRUNCATION Formulation

The most common formulation used to solve the controls-free, elastic aircraft, dynamics problem is MODAL TRUNCATION. This formulation is currently applied to the B-52 CCV (reference 11) and the B-1 (reference 29) aeroelastic analyses. This formulation reduces the computational steps required in the previous formulations by assuming A8 , and A9 below, in addition to A3 thru A7 assumed previously:

A8 the structural spring forces due to u_2 are zero, i.e., $k_2 u_2 = 0$.

A9 the aerodynamic forces due to u_2 are zero, i.e., $A_3 \phi_2 u_2 = 0$.

The application of assumptions A3 through A9 to equations (2.19) reduces them to the following from:

$$\begin{bmatrix} M & 0 & 0 \\ 0 & & 0 \\ 0 & & m_1 \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \ddot{r}_{op} \\ \ddot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM'_{1L} & 0 \\ 0 & 0 \\ 0 & \phi_1^T D \phi_1 \end{bmatrix} \begin{Bmatrix} \dot{r}_{op} \\ \dot{u}_1 \end{Bmatrix} + \begin{bmatrix} MM^2_{1L} & 0 \\ 0 & 0 \\ 0 & k_1 \end{bmatrix} \begin{Bmatrix} r_{op} \\ u_1 \end{Bmatrix} = \begin{Bmatrix} \bar{\Phi}_v f_i \\ \bar{\Phi}_r f_i \\ \phi_1^T f_i \end{Bmatrix} \tag{3.20a}$$

$$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 + f_c + f_g + f_+ \qquad (3.20b)$$

Inspection of these equations indicates all reference to $\phi_2 u_2$ has been deleted. This reduces the number of equations and unknowns to $6 + r$, where r is the number of retained modes ϕ_1.

This formulation has the advantage of a simplified aerodynamic representation. In addition, the required computer size and the computer central processing time are smaller than either residual formulation, thus allowing more analysis for the dollar.

The disadvantage of the formulation is that it has a narrower range of applicability than any of the other dynamic aeroelastic formulations. Thus, its use in the design of integrated flight control systems should be verified with quantitative and qualitative analyses that test the assumptions A3 through A9.

4.0 RESULTS

The primary results of this study are the six formulations of the equations that describe the small disturbance motions. These six formulations are summarized in Table 4.1. Here the similarity of the formulations is readily apparent as well as the reduction in the number of equations that occurs with each of the successive approximations. As shown in the table, the EXACT formulation, equations (2.19) are 3N+6 in number. The MODAL SUBSTITUTION formulation, equations (3.8), reduces the number of equations to 3N in number, while the RESIDUAL STIFFNESS, equations (3.13), RESIDUAL FLEXIBILITY, equations (3.19), and MODAL TRUNCATION, equations (3.20), formulations each consist of only 6+r equations where r is selected by the analyst. The value of r usually ranges from 2 to 30. The simplest formulation of the equations of motion is the QUASI STATIC, equations (3.4). It consists of only 6 equations and does not explicitly represent the elastic deformations.

The development costs of aircraft systems are often the major influence on the selection of the formulation of the equations of motion and, thus, costs must be weighed in any analysis as an additional deciding factor. In the event that two or more of the six formulations for the small distrubance motions are appropriate to a particular flight control analysis, a cost-effectiveness trade can be constructed similar to that presented in Table 4.2. As noted in the table, the increase in accuracy requires an increase in analysis cost due to increasing equation complexity. The reasons for this increase in cost and complexity are that both computer program run time and program size increase as the flight control analysis moves from the QUASI STATIC to the EXACT formulation. The primary reason for the increase in run time and cost is the necessity to employ highly accurate, computerized routines to calculate both the real and complex eigenvalues and eigenvectors of each system of equations. In addition, the RESIDUAL FLEXIBILITY formulation requires a non-standard flexibility matrix, the "free-free flexibility matrix", that is not required in other formulations. This flexibility matrix, related to the flexibility matrix of the cantilevered aircraft, must be recalculated, using the equation

$$\tilde{C} = (I - \bar{\Phi}\underline{M}^{-1}\bar{\Phi}^T m_i)C$$

each time the center of mass is changed, $\bar{\Phi}$, or the inertia of the aircraft, $\underline{M}$ and m_i, is altered. These two changes are common parametric studies in most flight control analyses and are particularly important in those studies that are used to design and analyze the reduced static stability augmentation systems.

FORMULATION	EQUATIONS OF MOTION	AERODYNAMIC FORCE
EXACT Eqs. (2.19)	$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ m_i\bar{\Phi}_v & m_i\bar{\Phi}_r & m_i \end{bmatrix}\begin{Bmatrix}\dot{v}\\ \ddot{r}_{op}\\ \ddot{q}_p\end{Bmatrix} + \begin{bmatrix} MM_{12}^1 & 0\\ 0 & 0\\ m_i\bar{\Phi}_v M_{12}^1 & D\end{bmatrix}\begin{Bmatrix}\dot{r}_{op}\\ \dot{q}_p\end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0\\ 0 & 0\\ m_i\bar{\Phi}_v M_{12}^2 & K\end{bmatrix}\begin{Bmatrix} r_{op}\\ q_p\end{Bmatrix} = \begin{Bmatrix}\bar{\Phi}_v^T f_i\\ \bar{\Phi}_r^T f_i\\ f_i\end{Bmatrix}$	$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 q_p + A_4 \dot{q}_p + A_5 \ddot{q}_p + f_c + f_g + f_t$
MODAL SUBSTITUTION Eqs. (3.8)	$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix}\dot{v}\\ \ddot{r}_{op}\\ \ddot{u}\end{Bmatrix} + \begin{bmatrix} MM_{12}^1 & 0\\ 0 & 0\\ 0 & \phi^T D\phi\end{bmatrix}\begin{Bmatrix}\dot{r}_{op}\\ \dot{u}\end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0\\ 0 & 0\\ 0 & k\end{bmatrix}\begin{Bmatrix} r_{op}\\ u\end{Bmatrix} = \begin{Bmatrix}\bar{\Phi}_v^T f_i\\ \bar{\Phi}_r^T f_i\\ \phi^T f_i\end{Bmatrix}$	$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 \phi u + A_4 \phi \dot{u} + A_5 \phi \ddot{u} + f_c + f_g + f_t$
RESIDUAL STIFFNESS Eqs. (3.13)	$\begin{bmatrix} M & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & m_1 \end{bmatrix}\begin{Bmatrix}\dot{v}\\ \ddot{r}_{op}\\ \ddot{u}_1\end{Bmatrix} + \begin{bmatrix} MM_{12}^1 & 0\\ 0 & 0\\ 0 & \phi_1^T D\phi_1\end{bmatrix}\begin{Bmatrix}\dot{r}_{op}\\ \dot{u}_1\end{Bmatrix} + \begin{bmatrix} MM_{12}^2 & 0\\ 0 & 0\\ 0 & k_1\end{bmatrix}\begin{Bmatrix} r_{op}\\ u_1\end{Bmatrix} = \begin{Bmatrix}\bar{\Phi}_v^T f_i\\ \bar{\Phi}_r^T f_i\\ \phi_1^T f_i\end{Bmatrix}$	$f_i = R_s^{-1}(A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 + f_c + f_g + f_t)$ $R_s = I - A_3 \phi_2 k_2^{-1} \phi_2^T$
RESIDUAL FLEXIBILITY Eqs. (3.19)	Same as RESIDUAL STIFFNESS	$f_i = R_F^{-1}(A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 + f_c + f_g + f_t + (A_3 R)(-m_i \phi_1 \ddot{u}_1 - D\phi_1 \dot{u}_1) - A_3 \bar{C} m_i \bar{\Phi}(\dot{v}_p + M_1 v_p + M_2 r_{op}'))$ $R_F = I - A_3 R$, $R = \bar{C} - \phi_1 k_1^{-1} \phi_1^T$
MODAL TRUNCATION Eqs. (3.20)	Same as RESIDUAL STIFFNESS	$f_i = A_1 v_p + A_2 \dot{v}_p + A_3 \phi_1 u_1 + A_4 \phi_1 \dot{u}_1 + A_5 \phi_1 \ddot{u}_1 + f_c + f_g + f_t$
QUASI STATIC Eqs. (3.4)	$\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix}\begin{Bmatrix}\dot{v}\\ \ddot{r}_{op}\end{Bmatrix} + \begin{bmatrix} MM_{12}^1 \\ 0 \end{bmatrix}\{\dot{r}_{op}\} + \begin{bmatrix} MM_{12}^2 \\ 0 \end{bmatrix}\{r_{op}\} = \begin{Bmatrix}\bar{\Phi}_v f_i\\ \bar{\Phi}_r f_i\end{Bmatrix}$	$f_i = Q\,(A_1 v_p + A_2 \dot{v}_p + f_c + f_g + f_t) + A_3 (K - A_3)^{-1}(-m_i \bar{\phi} \dot{v}_p - m_i \bar{\phi} H_1 v_p - m_i \bar{\phi} M_2 r_{op}')$ $Q = I + A_3 (K - A_3)^{-1}$

TABLE 4.1 SUMMARY OF SIX LINEAR FORMULATIONS OF THE EQUATIONS OF MOTION

METHOD	ECONOMICS		EIGENVALUE ROUTINE		STRUCTURAL DATA REQUIRED
	PROGRAM SIZE	PROGRAM RUN TIME	REAL NUMBER	COMPLEX NUMBER	
EXACT	LARGE	?	-	LARGE	[K]
MODAL SUBSTITUTION	LARGE	?	LARGE	LARGE	[K]
RESIDUAL STIFFNESS	MEDIUM	LONG	LARGE	MEDIUM	[K]
RESIDUAL FLEXIBILITY	MEDIUM	LONG	MEDIUM	MEDIUM	[K] AND [C]
MODAL TRUNCATION	MEDIUM	MEDIUM	MEDIUM	MEDIUM	[K]
QUASI STATIC	SMALL	SHORT	-	SMALL	[K]

TABLE 4.2 COST-EFFECTIVENESS OF SIX LINEAR FORMULATIONS OF THE EQUATIONS OF MOTION

5.0 CONCLUSIONS AND RECOMMENDATIONS

This paper has related the common formulations of the equations describing the motions of elastic aircraft. The conclusions of this study are that:

1. There is no formulation that is appropriate to all flight control analyses of elastic aircraft. The large disturbance formulation, equations (2.10), must be carefully specialized for each analysis due to their immense complexity. The small disturbance equations of motion that are most appropriate are those named the EXACT formulation, equations (2.19). However, in the interest of reducing analysis costs, the five approximate formulations of section 3.0 may also be appropriate. In no case should the QUASI STATIC, equations (3.4), or the MODAL TRUNCATION, equations (3.20), approximate formulations be chosen a priori.

2. The selection of an inappropriate formulation of the equations of motion can lead to large errors in the design of the flight control system. The actual numerical error cannot be determined at this time due to a lack of extensive numerical evaluations of the terms that are neglected to develop each formulation. The numerical evaluation of the significant terms neglected in each of the formulations should be a high priority research project. Contemporary aircraft should be used as the study configurations in this research.

3. The implementation of the equations of motion on flight simulators restricts the equations to the RESIDUAL STIFFNESS, equations (3.13), the RESIDUAL FLEXIBILITY, equations (3.19), the MODAL TRUNCATION, equations (3.20), or the QUASI STATIC, equations (3.4), formulations. This restriction occurs because the number of equations must be small such that the storage capacity of flight simulators (usually less than 40,000 central memory locations) is not exceeded. Obviously, the QUASI STATIC formulation, consisting of only 6 equations, is the simplest to use. Usually this formulation does not use all of the central memory of the flight simulator and, thus, the remainder of the central memory can be used to represent the nonlinear elements of the control system or the nonlinear aerodynamic effects of the elastic aircraft. However, the QUASI STATIC formulation does not represent the dynamics of the elastic structure and, for these analysis cases, the other three formulations must be applied.

4. The formulations that may most easily include the effects of nonlinear aerodynamics are the RESIDUAL STIFFNESS, RESIDUAL FLEXIBILITY, MODAL TRUNCATION, and QUASI STATIC formulations. This is because these four formulations can be reduced in size until central memory storage space in the digital or analog computer becomes available for the storage of the significant nonlinear aerodynamic data. The penalty for the reduction is, of course, a reduction in accuracy.

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